CS388S: Formal Semantics and Verification

Assignment 1: ω-Automaton and büchi acceptance

Due: 9/17/2015

Instructions: The solutions have to be typed (preferably in latex). However the figures can be hand drawn. You can work in groups of 2. Same grades are assigned to each member of the group. You can change groups between assignments. The deadlines are hard.

Definition 1. *Let $\mathcal{A} = (Q, \Sigma, \delta, q_I, Acc)$ be an ω-automaton. A run of $\mathcal{A}$ on an ω-word $\alpha = a_1a_2\ldots \in \Sigma^\omega$ is an infinite state sequence $\varrho = \varrho(0)\varrho(1)\ldots \in Q^\omega$ such that the following conditions hold:*

1. $\varrho(0) = q_I$
2. $\varrho(i) \in \delta(\varrho(i-1), a_i)$ *for* $i \geq 1$ *if* $\mathcal{A}$ *is non-deterministic,*
   $\varrho(i) = \delta(\varrho(i-1), a_i)$ *for* $i \geq 1$ *if* $\mathcal{A}$ *is deterministic.*

Büchi Acceptance: An ω-automaton with acceptance component $F \subseteq Q$. A word $\alpha \in \Sigma^\omega$ is accepted by $\mathcal{A}$ iff there exists a run ϱ of $\mathcal{A}$ on α satisfying the condition:

$$Inf(\varrho) \cap F \neq \emptyset$$

The language accepted by the automaton $\mathcal{A}$ is $L(\mathcal{A}) = \{\alpha \in \Sigma^\omega | \mathcal{A} \text{ accepts } \alpha\}$.

1. *In ω-automaton with the büchi acceptance prove that non-determinism is strictly more expressive than determinism. (That is there exists a language L_1 which can be recognized by a non-deterministic büchi automaton, but not by any deterministic büchi automaton.)*

ω**-regular:** A language L is said to be ω-regular if it is accepted by some büchi automaton.

Definition 2 (Limit Languages)**.** *Let L be a finite-regular language. Define $\hat{L}$ to be the ω-language consisting of all the words that have infinitely many prefixes in L.*

2. *Let $\mathcal{A}$ be a deterministic büchi automaton and let $L_f(\mathcal{A})$ be the language of finite words accepted by $\mathcal{A}$ when treated as a finite automaton and let $L(\mathcal{A})$ be the language accepted by $\mathcal{A}$ as a büchi automaton. Prove that*

$$L(\mathcal{A}) = \widehat{L_f(\mathcal{A})}.$$

3. *Show that the emptiness problem is decidable for büchi automata.*

The following set of problems will help you prove that ω-regular languages are closed under union, intersection, complementation.

4. *Show how to construct a büchi automaton accepting $L_1 \cap L_2$ given automaton accepting L_1 and L_2.*

5. *Show how to construct a deterministic büchi automaton accepting $L_1 \cup L_2$ and $L_1 \cap L_2$ from deterministic automaton accepting L_1 and L_2.*

Intersection and union are relatively easy. Next we show that ω-regular languages are closed under complementation.

For the complementation of regular languages Myhill-Nerode congruences was one of the tools available to us. We establish a similar notion of congruences to capture accepting runs over infinite words.

Definition 3. *With each büchi automaton $\mathcal{A}$, we associate a relation $\equiv_{\mathcal{A}}$ over Σ^* given by*

$$x \equiv_{\mathcal{A}} y \triangleq \forall q, q'.\ q \xrightarrow{x} q' \Leftrightarrow q \xrightarrow{y} q' \wedge$$
$$\forall q, q'. q \xrightarrow{x}_g q' \Leftrightarrow q \xrightarrow{y}_g q'$$

where $q \xrightarrow{x}_g q'$ means that there is run from q to q' on the word x that passes through some state in F. [1]

6. *Show that $\equiv_{\mathcal{A}}$ is a congruence.*

7. *Let U and V be regular languages. Then show that $U.V^\omega$ is ω-regular.*

From (6) and (7) if $\equiv_{\mathcal{A}}$ has N equivalence classes, then each of the ω-language obtained as $[u].[v]^\omega$ is either completely contained in $L(\mathcal{A})$ or in $\overline{L(\mathcal{A})}$. Unlike the case of finite words where it is trivial that each word of Σ^* lies in some equivalence class of $\equiv_{\mathcal{A}}$, it is not clear that every ω-word is an element of $[u][v]^\omega$ for some u and v. This needs proof. First prove a general result about monoids.

8. *Let F be the free monoid* [2] *over Σ (possibly infinite). M be a finite monoid and let h be a homomorphism from F to M. Let $\alpha = a_1 a_2 \dots$ be any ω-word over Σ. Then, there are elements s and e in M such that $e.e = e$ and $\alpha \in h^{-1}(s).(h^{-1}(e))^\omega$.*

(Hint: Use the infinite version of the Ramsey's theorem. However it is also possible to give a direct argument.)

9. *Use (8) and $\Sigma_{\mathcal{A}}$ to find the complement of an ω-language L. (Hint: Use the quotient monoid.)*

10. *Prove that a language L over Σ^ω is ω-regular iff there is a homomorphism h from $\Sigma*$ to a finite monoid M and a collection X of linked pairs (Pairs (s, e) such that $s.e = s$ and $e.e = e$) over M such that $L = \bigcup_{(s,e)\in X} h^{-1}(s).(h^{-1}(e))^\omega$.*

[1] I shall write $[x]$ for $[x]_{\equiv_{\mathcal{A}}}$.

[2] For a definition of free monoid on Σ refer to Automata and Computability by Dexter Kozen.

References

Dominique Perrin and Jean-Eric Pin : Infinite words: Automata, Semi-groups, Logic and Games LNCS 2500

Madhavan Mukund: (Lecture Notes) Finite Automata on Infinte words

Wolfgang Thomas: Automata over Infinite words