Knaster-Tarski Theorem
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9/12/2014

This note presents a proof of the famous Knaster-Tarski theorem [1]. I have opted for clarity over brevity in the proof.

Complete Lattice A *complete lattice* is a partially ordered set $(L, \leq)$ in which every subset of L has a greatest lower bound (glb) and a least upper bound(lub) in L. The lub of the empty set is denoted by $\bot$ and glb of the empty set is $\top$. It follows that a complete lattice is never empty.[1] Henceforth, *point* is an element of L.

Monotonic Function over a Lattice Let f be a monotonic function over a lattice $(L, \leq)$, i,e, $x \leq y \;\Rightarrow\; f(x) \leq f(y)$, for any x and y in L. Point x is a (1) *prefixed* point if $f(x) \leq x$, (2) *postfixed* point if $x \leq f(x)$, and (3) *fixed* point if $f(x) = x$. Clearly, a fixed point is both a prefixed and a postfixed point.

Theorem [Knaster-Tarski]: For any complete lattice $(L, \leq)$,

1. The least fixed and the prefixed points of f exist, and they are identical.
2. The greatest fixed and the postfixed points of f exist, and they are identical.
3. The fixed points form a complete lattice.

Proofs of (1, 2) Proofs of (1) and (2) are duals and we prove only (1). Let *pre* be the set of prefixed points, and p the glb of *pre*. Existence of p is guaranteed since L is a complete lattice. We show that p is both the least prefixed point and the least fixed point.

• p is the least prefixed point: For any prefixed point x,

$p \leq x$
$\Rightarrow$ {f is monotonic}
$f(p) \leq f(x)$
$\Rightarrow$ {x is a prefixed point; so, $f(x) \leq x$}
$f(p) \leq x$
$\Rightarrow$ {from above, $f(p)$ is a lower bound of *pre*. And, p is the glb of *pre*.}
$f(p) \leq p$
$\Rightarrow$ {from above, p is a prefixed point; and also a lower bound of *pre*.}
p is the least prefixed point

[1] A lattice with a top and a bottom element is not necessarily a complete lattice. Here is a counterexample due to Vladimir Lifschitz. Let R be the set of rational numbers in the closed interval [0,1]; clearly R is a lattice under the standard order with a top and a bottom. For any irrational number x between 0 and 1 let I_x be the set of numbers in R that are less than x. Then I_x does not have a lub in R.

- p is the least fixed point: Since p is a prefixed point,

$$
\begin{array}{ll}
 & f(p) \leq p \\
\Rightarrow & \{f \text{ is monotonic}\} \\
 & f(f(p)) \leq f(p) \\
\Rightarrow & \{\text{from above, } f(p) \text{ is a prefixed point; and } p \text{ is a lower bound over } pre.\} \\
 & p \leq f(p) \\
\Rightarrow & \{p \text{ is a prefixed point; so } f(p) \leq p\} \\
 & p = f(p) \\
\Rightarrow & \{ \text{ Every fixed point is a prefixed point.} \\
 & \quad p \text{ is a lower bound over all prefixed points, so over all fixed points.}\} \\
 & p \text{ is a fixed point and a lower bound over all fixed points} \\
\Rightarrow & \{\text{simple reasoning}\} \\
 & p \text{ is the least fixed point}
\end{array}
$$

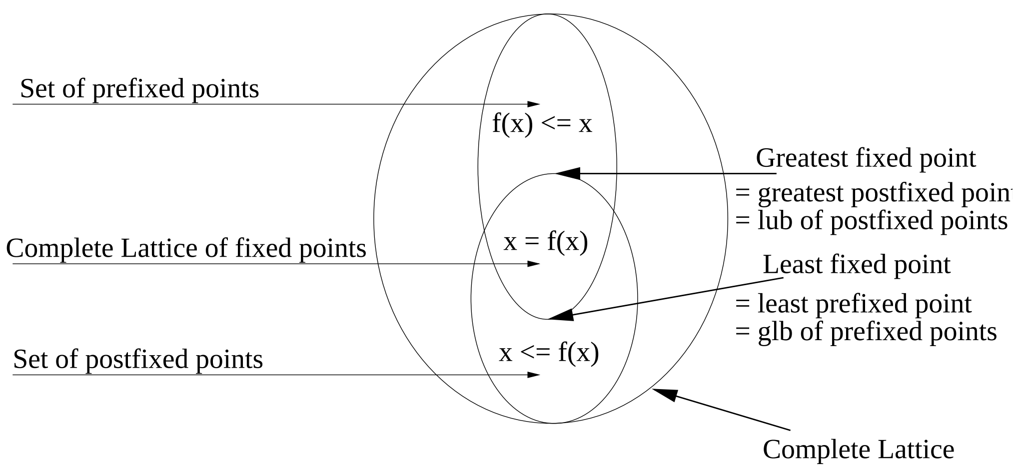

Figure 1: Pictorial Depiction of the Knaster-Tarski Theorem

Proof of (3), The fixed points form a complete lattice: Let W be a subset of the fixed points. We show the existence of the supremum of W. Dually, W has an infimum, establishing that the fixed points form a complete lattice.

Let $q = lub(W)$ and $\widehat{W} = \{w \mid q \le w\}$. Then $q \in \widehat{W}$ and $q = glb(\widehat{W})$.

3.1 $\widehat{W}$ is a complete lattice: $\widehat{W}$, being a subset of a complete lattice, has a lub and glb. We have $q = glb(\widehat{W})$ and $q \in \widehat{W}$; so, $glb(\widehat{W}) \in \widehat{W}$. Further, since $q \in \widehat{W}$, $q \le lub(\widehat{W})$; therefore, $lub(\widehat{W}) \in \widehat{W}$ from the definition of $\widehat{W}$.

3.2 f maps $\widehat{W}$ to $\widehat{W}$: For any element w of W and x of $\widehat{W}$

$$
\begin{array}{ll}
 & w \le q \text{ and } q \le x \\
\Rightarrow & \{f \text{ monotonic: so } f(w) \le f(q) \le f(x)\} \\
 & f(w) \le f(x) \\
\Rightarrow & \{W \text{ is a set of fixed points; so, } f(w) = w\} \\
 & w \le f(x) \\
\Rightarrow & \{w \text{ is an arbitrary element of } W\} \\
 & lub(W) \le f(x) \\
\Rightarrow & \{q = lub(W)\} \\
 & q \le f(x) \\
\Rightarrow & \{\text{definition of } \widehat{W}\} \\
 & f(x) \in \widehat{W}
\end{array}
$$

3.3 From (3.1 and 3.2), f is a mapping over the complete lattice $\widehat{W}$. So, it has a least fixed point $\widehat{q}$ in $\widehat{W}$. Since q is the least element of $\widehat{W}$, $q \le \widehat{q}$. Thus, $\widehat{q}$ is the supremum of W.

References

[1] A. Tarski. A lattice-theoretical fixpoint theorem and its application. *Pacific Journal of Mathematics*, 5:285–309, 1955.