CTL Model Checking

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Formal Systems II

Fixed Point Theory

Fixed Points Definition
Definition

Let $f : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$ be a set valued function and Z a subset of G.

1. Z is called a *fixed point* of f if $f(Z) = Z$.
2. Z is called the *least fixed point* of f is Z is a fixed point and for all other fixed points U of f the relation $Z \subseteq U$ is true.
3. Z is called the *greatest fixed point* of f is Z is a fixed point and for all other fixed points U of f the relation $U \subseteq Z$ is true.

Finite Fixed Point Lemma

Let G be an arbitrary set, Let $\mathcal{P}(G)$ denote the *power set* of a set G.
A function $f : \mathcal{P}(G) \to \mathcal{P}(G)$ is called monotone of for all $X, Y \subseteq G$

$$X \subseteq Y \Rightarrow f(X) \subseteq f(Y)$$

Let $f : \mathcal{P}(G) \to \mathcal{P}(G)$ be a monotone function on a *finite* set G.

1. There is a least and a greatest fixed point of f.
2. $\bigcup_{n \geq 1} f^n(\emptyset)$ is the least fixed point of f.
3. $\bigcap_{n \geq 1} f^n(G)$ is the greatest fixed point of f.

Proof

of part (2) of the finite fixed point Lemma

Monotonicity of f yields

$$\emptyset \subseteq f(\emptyset) \subseteq f^2(\emptyset) \subseteq \ldots \subseteq f^n(\emptyset) \subseteq \ldots$$

Since G is finite there must be an i such that $f^i(\emptyset) = f^{i+1}(\emptyset)$.
Then $Z = \bigcup_{n \geq 1} f^n(\emptyset) = f^i(\emptyset)$ is a fixed point of f:

$$f(Z) = f(f^i(\emptyset)) = f^{i+1}(\emptyset) = f^i(\emptyset) = Z$$

Let U be another fixed point of f.
From $\emptyset \subseteq U$ be infer by monotonicity of f at first $f(\emptyset) \subseteq f(U) = U$.
By induction on n we conclude $f^n(\emptyset) \subseteq U$ for all n.
Thus also $Z = f^i(\emptyset) \subseteq U$.

Continuous Functions

Definition

A function $f : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$ is called

1. $\cup$-**continuous** (upward continuous), if for every ascending sequence $M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots$

$$f(\bigcup_{n \geq 1} M_n) = \bigcup_{n \geq 1} f(M_n)$$

2. $\cap$-**continuous** (downward continuous) , if for every descending sequence $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$

$$f(\bigcap_{n \geq 1} M_n) = \bigcap_{n \geq 1} f(M_n)$$

Fixed Points For Continuous Functions

Let
$f : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$ be an upward continuous functions and
$g : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$ a downward continuous function.

The for all $M, N \in \mathcal{P}(G)$ such that $M \subseteq f(M)$ and $g(N) \subseteq N$ the following is true.

1. $\bigcup_{n \geq 1} f^n(M)$ is the least fixed point of f containing M,
2. $\bigcap_{n \geq 1} g^n(M)$ is the greatest fixed point of g contained in M.

Proof of (1)

By monotonicity we first obtain

$$M \subseteq f(M) \subseteq f^2(M) \subseteq \ldots \subseteq f^n(M) \subseteq \ldots$$

Let $P = \bigcup_{n \geq 1} f^n(M)$. This immediately gives $M \subseteq P$. Furthermore

$$\begin{array}{rcll} f(P) & = & f(\bigcup_{n \geq 1} f^n(M)) & \\ & = & \bigcup_{n \geq 1} f^{n+1}(M) & \text{by continuity} \\ & = & \bigcup_{n \geq 1} f^n(M) & \text{since } f(M) \subseteq f^2(M) \\ & = & P & \end{array}$$

Assume now that Q is another fixed point of f satisfying $M \subseteq Q$.
By Monotonicity and the fixed point property $f(M) \subseteq f(Q) = Q$ and furthermore for every $n \geq 1$ also $f^n(M) \subseteq Q$.
Thus we obtain $P = \bigcup_{n \geq 1} f^n(M) \subseteq Q$

Knaster-Tarski-Fixed-Points Theorem

Let $f : \mathcal{P}(G) \to \mathcal{P}(G)$ be a monotone function. Then

$$f \text{ has a least and a greatest fixed point.}$$

Note, G need not be finite.

Proof

Fixed Point

Let $L = \{S \subseteq G \mid f(S) \subseteq S\}$, e.g., $G \in L$.
Let $U = \bigcap L$. Show $f(U) = U$!

For all $S \in L$ by the property of an intersection: $U \subseteq S$.
By monotonicity and definition of L $f(U) \subseteq f(S) \subseteq S$.
Thus $f(U) \subseteq \bigcap L = U$ and we have already established half of our claim.

By monotonicity $f(U) \subseteq U$ implies $f(f(U)) \subseteq f(U)$
which yields $f(U) \in L$ and futhermore $U \subseteq f(U)$.

We thus have indeed $U = f(U)$.

Proof

Least Fixed Point

Now assume W is another fixed point of f,i.e., $f(W) = W$.
This yields $W \in L = \{S \subseteq G \mid f(S) \subseteq S\}$ and
$U = \bigcap L \subseteq W$.
Thus U is the least fixed point of f.

Following the same line of argument one can show that
$\bigcup\{S \subseteq G \mid S \subseteq f(S)\}$ is the greatest fixed point of f.

CTL Model Checking

Algorithm

Two Auxiliary Concepts

Let $\mathcal{T} = (S, R, v)$ be a transition system and F an CTL formula.
The set

$$\tau(F) = \{s \in S \mid s \models F\}$$

is called the characteristic region of F in $\mathcal{T}$.

The universal and existential next step functions
$f_{AX}, f_{EX} : \mathcal{P}(S) \to \mathcal{P}(S)$ are defined by

$$\begin{array}{lcl} f_{AX}(Z) & = & \{s \in S \mid \text{for all } t \text{ with } sRt \text{ we get } t \in Z\} \\ f_{EX}(Z) & = & \{s \in S \mid \text{there exists a } t \text{ with } sRt \text{ and } t \in Z\} \end{array}$$

CTL Model Checking Algorithm

Let $\mathcal{T} = (S, R, v)$ be a transition system and F an CTL formula. $\tau(F)$ is computed by the following high-level recursive algorithm:

1	$\tau(p)$	$=$	$\{s \in S \mid v(s,p) = \mathbf{1}\}$
2	$\tau(F_1 \wedge F_2)$	$=$	$\tau(F_1) \cap \tau)F_2)$
3	$\tau(F_1 \vee F_2)$	$=$	$\tau(F_1) \cup \tau(F_2)$
4	$\tau(\neg F_1)$	$=$	$S \setminus \tau(F_1)$
5	$\tau(\mathbf{A}(F_1\ \mathbf{U}\ F_2))$	$=$	$\mathrm{lfp}[\tau(F_2) \cup (\tau(F_1) \cap f_{AX}(Z))]$
6	$\tau(\mathbf{E}(F_1\ \mathbf{U}\ F_2))$	$=$	$\mathrm{lfp}[\tau(F_2) \cup (\tau(F_1) \cap f_{EX}(Z)]$
7	$\tau(\mathbf{AF}F_1)$	$=$	$\mathrm{lfp}[\tau(F_1) \cup f_{AX}(Z)]$
8	$\tau(\mathbf{EF}F_1)$	$=$	$\mathrm{lfp}[\tau(F_1) \cup f_{EX}(Z)]$
9	$\tau(\mathbf{EG}F_1)$	$=$	$\mathrm{gfp}[\tau(F_1) \cap f_{EX}(Z)]$
10	$\tau(\mathbf{AG}F_1)$	$=$	$\mathrm{gfp}[\tau(F_1) \cap f_{AX}(Z)]$

Correctness Proof

Case 10 $\mathbf{AG}\ \mathbf{F_1}$

By definition

$$\tau(\mathbf{AG}\ F_1) = \{s \in S \mid s \in \tau(F_1) \text{ and } h \in \tau(\mathbf{AG}\ F_1) \text{ for all } h \text{ with } gRh\}$$

Using Definition of the universal next step function:

$$\tau(\mathbf{AG}\ F_1) = \tau(F_1) \cap f_{AX}(\tau(\mathbf{AG}\ F_1))$$

So, $\tau(\mathbf{AG}\ F_1)$ is a fixed point of the function $\tau(F_1) \cap f_{AX}(Z)$.

It remains to see that it is the greatest fixed point.

Correctness Proof

Case 10 $\mathbf{AG\ F_1}$ (continued)

Let H be another fixed point, i.e., $H = \tau(F_1) \cap f_{AX}(H)$.
We need to show $H \subseteq \tau(\mathbf{AG}\ F_1))$.
Consider $g_0 \in H$ with the aim of showing that for all $n \geq 0$ and all g_i satisfying $g_{i-1} R g_i$ for all $1 \leq i \leq n$ we obtain $g_n \in \tau(F_1)$.

Observe $\quad H = \tau(F_1) \cap f_{AX}(H) \subseteq \tau(F_1)$.

Thus it suffices to show for all $n \geq 0$ that $g_n \in H$.

For $n = 0$ that is true by assumptions.

Assume $g_{n-1} \in H$.
$g_{n-1} \in H \subseteq f_{AX}(H) = \{g \mid \text{ for all } h \text{ with } gRh \text{ we have } h \in H\}$.

Thus $g_n \in H$.

Correctness Proof

Case 6 $\mathbf{E(F_1\ U\ F_2)}$

By definition

$$\begin{array}{rcl} \tau(\mathbf{E}(F_1\ \mathbf{U}\ F_2)) & = & \{g \in S \mid s \models F_2 \text{ or } s \models F_1 \text{ and} \\ & & \quad \text{there exists } h \text{ with } gRh \text{ and } h \models F_2\} \end{array}$$

Using next step function:

$$\tau(\mathbf{E}(F_1\ \mathbf{U}\ F_2)) = \tau(F_2) \cup (\tau(F_1) \cap f_{EX}(\tau(\mathbf{E}(F_1\ \mathbf{U}\ F_2)))$$

Thus $\tau(\mathbf{E}(F_1\ \mathbf{U}\ F_2))$ is a fixed point of the function

$$\tau(F_2) \cup (\tau(F_1) \cap f_{EX}(Z)).$$

It remains to show that it is the least fixed point.

Correctness Proof

Case 6 $\mathbf{E}(\mathbf{F_1}\ \mathbf{U}\ \mathbf{F_2})$ (continued)

Consider $H \subseteq S$ with $H = \tau(F_2) \cup (\tau(F_1) \cap f_{EX}(H))$.

We need $\tau(\mathbf{E}(F_1\ \mathbf{U}\ F_2)) \subseteq H$.

Fix $g_0 \in \tau(\mathbf{E}(F_1\ \mathbf{U}\ F_2))$, try to arrive at $g_0 \in H$.

There is an $n \in \mathbb{N}$ and there are g_i for $1 \leq i \leq n$ satisfying

1. $g_i R g_{i+1}$ for all $0 \leq i < n$.
2. $g_n \in \tau(F_2)$.
3. $g_i \in \tau(F_1)$ for all $0 \leq i < n$.

We set out to prove $g_n \in H$ by induction on n.

$\mathbf{n} = 0$ Since $H = \tau(F_2) \cup (\tau(F_1) \cap f_{EX}(H)) \supseteq \tau(F_2)$ and $g_0 = g_n \in \tau(F_2)$.

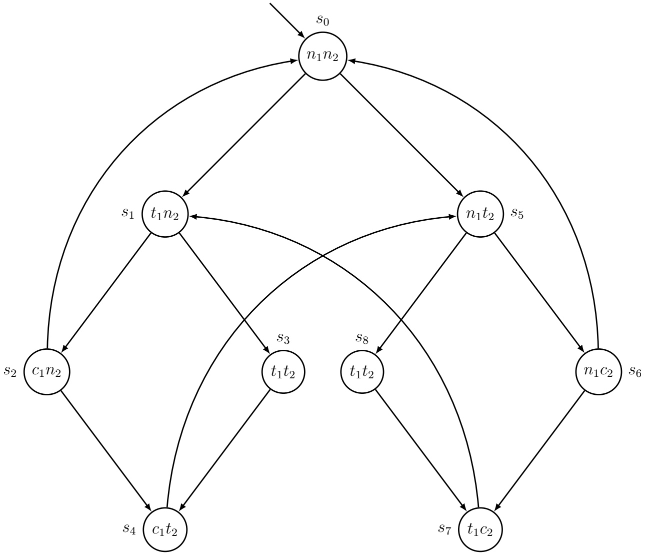
$\mathbf{n-1} \rightsquigarrow \mathbf{n}$ By induction hypothesis we have $g_1 \in H$

Since $g_0 \in \tau(F_1)$ and $g_0 R g_1$ we obtain $g_0 \in (\tau(F_1) \cap f_{EX}(H)) \subseteq H$ and thus also $g_0 \in H$.

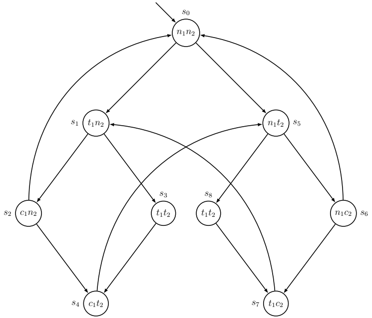
CTL Model Checking

Example

Transition System



The Task



	0	1	2	3	4	5	6	7	8
N_1	1	0	0	0	0	1	1	0	0
N_2	1	1	1	0	0	0	0	0	0
T_1	0	1	0	1	0	0	0	1	1
T_2	0	0	0	1	1	1	0	0	1
C_1	0	0	1	0	1	0	0	0	0
C_2	0	0	0	0	0	0	1	1	0

Check if the following the formula is true in state 1.

$$F = T_1 \rightarrow \mathbf{AF}C_1 \equiv \neg T_1 \vee \mathbf{AF}C_1$$

The Set-up

Goal

$$1 \models T_1 \rightarrow \mathbf{AF}C_1 \quad \text{i.e.} \quad 1 \models \neg T_1 \vee \mathbf{AF}C_1$$

We will present the computations starting with the innermost subformulas first, i.e., in the order

$$\tau(T_1),\ \tau(C_1),\ \tau(\neg T_1),\ \tau(\mathbf{AF}C_1), \text{ and } \tau(F).$$

This will make it much easier to follow the algorithm, since when a recursice call is started, we know already its result.

In the end we check $1 \in \tau(F)$.

The First Steps

$$\tau(T_1) = \{1, 3, 7\} \tag{1}$$

$$\tau(C_1) = \{2, 4\} \tag{2}$$

From which we get easily

$$\tau(\neg T_1) = \{0, 2, 4, 5, 6\} \tag{3}$$

Fixed Point Computation

The next step is to compute $\tau(\mathbf{AF}C_1)$
according to the algorithm this amounts to the computation of the least fixed point of $f(Z) = \tau(C_1) \cup f_{AX}(Z) = f_{AX}(Z) \cup \{2, 4\}$.
We thus compute $f(\emptyset)$, $f^2(\emptyset)$, ... $f^n(\emptyset)$ till we reach a stationary value, i.e. $f^n(\emptyset) = f^{n+1}(\emptyset)$.

$$\begin{array}{rcl}
f^1(\emptyset) & = & \{2, 4\} \\
f^2(\emptyset) & = & \{2, 3, 4\} \\
f^3(\emptyset) & = & \{1, 2, 3, 4\} \\
f^4(\emptyset) & = & \{1, 2, 3, 4, 7\} \\
f^5(\emptyset) & = & \{1, 2, 3, 4, 7, 8\} \\
f^6(\emptyset) & = & \{1, 2, 3, 4, 7, 8\}
\end{array}$$

Thus

$$\tau(\mathbf{AF}C_1) = \{1, 2, 3, 4, 7, 8\} \tag{4}$$

End of the Computation

$$\tau(F) = \tau(\neg T_1) \cup \tau(\mathbf{AF}C_1) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\} = S \tag{5}$$

Since $1 \in \tau(F)$ we conclude

$$s_1 \models F \tag{6}$$

THE

END