Recall: the one-time pad is not reusable (i.e., the two-time pad is totally broken)

NEVER REUSE THE KEY TO A STREAM CIPHER!

But wait... we "proved" that a stream cipher was secure, and yet, there is an attack?

Recall security game:

$b \in \{0,1\}$

| adversary | | challenger |
|---|---|---|
| | $m_0, m_1 \longrightarrow$ | $k \xleftarrow{R} \mathcal{K}$ |
| | $\longleftarrow c_b$ | $c_b \leftarrow \text{Encrypt}(k, m_b)$ |
| $b' \in \{0,1\}$ | | |

Observe: adversary only sees one ciphertext
key is only used once

$\Rightarrow$ Security in this model says nothing about multiple messages / ciphertexts

Problem: If we want security with multiple ciphertexts, we need a different or stronger definition (CPA security)

**Definition**: An encryption scheme $\Pi_{SE} = (\text{Encrypt}, \text{Decrypt})$ is secure against chosen-plaintext attacks (CPA-secure) if for all efficient adversaries $\mathcal{A}$:

$$\text{CPAAdv}[\mathcal{A}, \Pi_{SE}] = \left| \Pr[W_0 = 1] - \Pr[W_1 = 1] \right| = \text{negl.}$$

where $W_b$ ($b \in \{0,1\}$) is the output of the following experiment:

$b \in \{0,1\}$

adversary | challenger: $k \xleftarrow{R} \mathcal{K}$

$m_0, m_1 \in \mathcal{M}$ →

← $\text{Encrypt}(k, m_b)$

$b' \in \{0,1\}$

← same idea as in original semantic security game, but allow adversary to make encryption queries (also called a "left-or-right" oracle)

output of experiment $W_b$

[Adversary's goal is to guess which of $m_0$ or $m_1$ was encrypted, given access to an encryption oracle (i.e., adversary gets to see encryptions of messages of its choice.]

**Claim.** A stream cipher is not CPA-secure.

**Proof.** Consider the following adversary:

$b \in \{0,1\}$

adversary | challenger

choose $m_0, m_1 \in \mathcal{M}$ where $m_0 \neq m_1$ | $s \xleftarrow{R} \{0,1\}^\lambda$

$m_0, m_0$ →

← $c = m_0 \oplus G(s)$

$m_0, m_1$ →

← $c' = m_b \oplus G(s)$

output 0 if $c = c'$
output 1 if $c \neq c'$

$\Pr[b' = 1 \mid b = 0] = 0$ since $c' = m_0 \oplus G(s) = c$

$\Pr[b' = 1 \mid b = 1] = 1$ since $c' = m_1 \oplus G(s) \neq c$

$\Rightarrow \text{CPAAdv}[\mathcal{A}, \Pi_{SE}] = 1$

**Observe**: Above attack works for any deterministic encryption scheme.

$\Rightarrow$ CPA-secure encryption must be randomized!

$\Rightarrow$ To be reusable, cannot be deterministic. Encrypting the same message twice should not reveal that identical messages were encrypted.

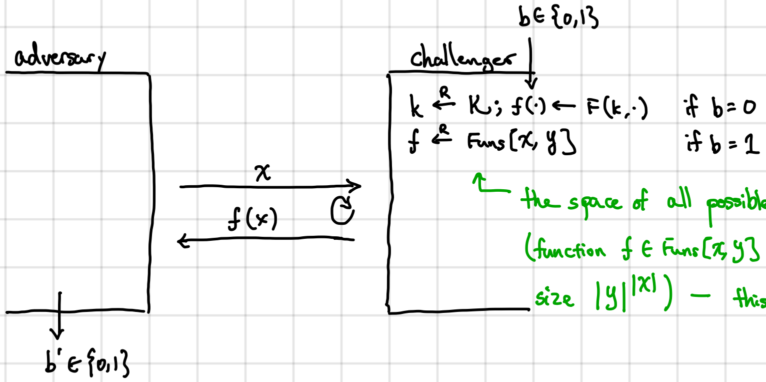
To build a CPA-secure encryption scheme, we will use a "block cipher"

- Block cipher is an invertible keyed function that takes a block of $n$ input bits and produces a block of $n$ output bits
- Examples include 3DES (key size 168 bits, block size 64 bits)
  AES (key size 128 bits, block size 128 bits)

Will define block ciphers abstractly first: pseudorandom functions (PRFs) and $\overbrace{\text{pseudorandom permutations (PRPs)}}^{\text{block ciphers}}$

↳ General idea: PRFs behave like random functions
PRPs behave like random permutations

**Definition.** A function $F: \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$ with key-space $\mathcal{K}$, domain $\mathcal{X}$, and range $\mathcal{Y}$ is a pseudorandom function (PRF) if for all efficient adversaries $\mathcal{A}$, $|W_0 - W_1| = \text{negl.}$, where $W_b$ is the probability the adversary outputs 1 in the following experiment:




the space of all possible functions from $\mathcal{X} \to \mathcal{Y}$
(function $f \in \text{Funs}[\mathcal{X}, \mathcal{Y}]$ can be represented by a truth table of size $|\mathcal{Y}|^{|\mathcal{X}|}$) — this is usually exponentially large!

$$\text{PRFAdv}[\mathcal{A}, F] = |W_0 - W_1| = |\Pr[\mathcal{A} \text{ outputs } 1 \mid b = 0] - \Pr[\mathcal{A} \text{ outputs } 1 \mid b = 1]|$$

**Intuitively:** input-output behavior of a PRF is indistinguishable from that of a random function (to any computationally-bounded adversary)

3DES: $\{0,1\}^{168} \times \{0,1\}^{64} \to \{0,1\}^{64}$ $\quad |\mathcal{K}| = 2^{168}$ $\quad |\text{Funs}[\mathcal{X}, \mathcal{Y}]| = (2^{64})^{(2^{64})}$

AES: $\{0,1\}^{128} \times \{0,1\}^{128} \to \{0,1\}^{128}$ $\quad |\mathcal{K}| = 2^{128}$ $\quad |\text{Funs}[\mathcal{X}, \mathcal{Y}]| = (2^{128})^{(2^{128})}$

} space of random functions is exponentially-larger than key-space!

**Definition:** A function $F: \mathcal{K} \times \mathcal{X} \to \mathcal{X}$ is a pseudorandom permutation (PRP) if

- for all keys $k$, $F(k, \cdot)$ is a permutation and moreover, there exists an efficient algorithm to compute $F^{-1}(k, \cdot)$:

$$\forall k \in \mathcal{K} : \forall x \in \mathcal{X} : F^{-1}(k, F(k, x)) = x$$

- for $k \xleftarrow{R} \mathcal{K}$, the input-output behavior of $F(k, \cdot)$ is computationally indistinguishable from $f(\cdot)$ where $f \xleftarrow{R} \text{Perm}[\mathcal{X}]$ and $\text{Perm}[\mathcal{X}]$ is the set of all permutations on $\mathcal{X}$ (analogous to PRF security)

**Note:** a block cipher is another term for PRP (just like stream ciphers are PRGs)

Observe that a block cipher can be used to construct a PRG:

$F : \{0,1\}^\lambda \times \{0,1\}^n \to \{0,1\}^n$ be a block cipher

Define $G : \{0,1\}^\lambda \to \{0,1\}^{\ell n}$ as

$$G(k) = F(k,1) \| F(k,2) \| \cdots \| F(k,\ell)$$

← this stream cipher allows <u>random</u> access!

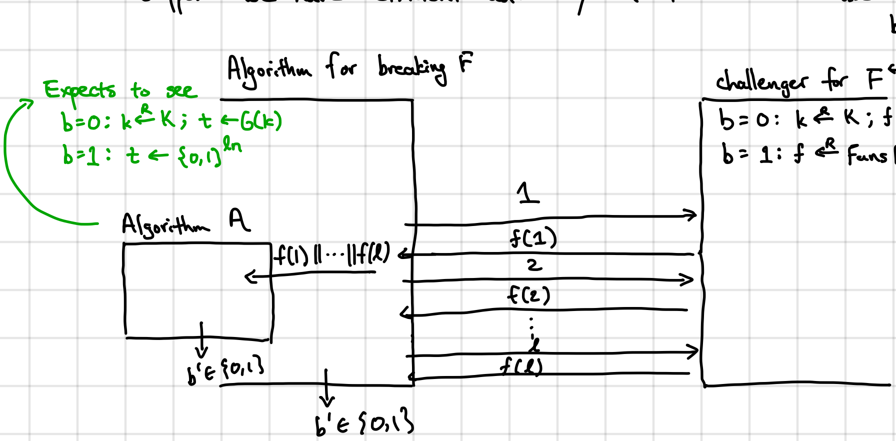
↑ string concatenation ↑ write input as an $n$-bit string (just require that $n > \log \ell$)

we said PRP above (will revisit this)

<u>Theorem.</u> If $F$ is a secure PRF, then $G$ is a secure PRG.

<u>Proof.</u> As usual, we show the contrapositive: if $G$ is not a secure PRG, then $F$ is not a secure PRF.

Suppose we have efficient adversary $A$ for $G$. We use $A$ to build adversary for $F$:




1. If $\ell = \text{poly}$, then $B$ is efficient
2. If $b = 0$: $B$ sends $G(k)$ to $A$ where $k$ is a uniformly random key

   If $b = 1$: $B$ sends uniformly random string ($f$ is <u>random</u> function) to $A$
3. $\text{PRFAdv}[B,F] = |\Pr[b'=1 \mid b=0] - \Pr[b'=1 \mid b=1]|$

$= |\Pr[A \text{ outputs } 1 \mid b=0] - \Pr[A \text{ outputs } 1 \mid b=1]|$

$= \text{PRGAdv}[A,G]$

which is non-negligible by assumption.

But... we used a block cipher (PRP) in our construction above. Does the proof still go through?

Not quite...

for a random function, $f(1) = f(2)$ with probability $\frac{1}{2^n}$

for a random permutation, $f(1) = f(2)$ with probability $0$

} but $2^{-n}$ might be very very small... adversary won't notice unless it sees a "collision" [i.e., two values $x, y$ where $f(x) = f(y)$]

<u>PRF Switching Lemma.</u> Let $F : K \times X \to X$ be a secure PRP. Then, for any $Q$-query adversary $A$:

$$|\text{PRPAdv}[A,F] - \text{PRFAdv}[A,F]| \leq \frac{Q^2}{2|X|}$$

<u>Proof Idea.</u> Adversary essentially cannot tell the difference unless it sees a collision. If there is no collision, then it is just seeing random values. How many queries before there is a collision? Birthday paradox: $Q \sim \sqrt{|X|}$

<u>Take-away</u>: If $|X|$ is large (e.g., exponential), then we can use a PRP as a PRF.

- 3DES: $n = 64$ so $|X| = 2^{64}$ [if adversary makes $\ll 2^{32}$ queries, then can use it as a PRF]
- AES: $n = 128$ so $|X| = 2^{128}$ [if adversary makes $\ll 2^{64}$ queries, then can use it as a PRF]