So far in this course: assumption is that adversary is classical.
How do things change if adversaries are quantum? We won't go into detail but will state main results:

Grover's algorithm: Given black-box access to a function $f : [N] \to \{0,1\}$, Grover's algorithm finds an $x \in [N]$ such that $f(x) = 1$ by making $O(\sqrt{N})$ queries to f.

"Searching an unsorted database of size N in time $O(\sqrt{N})$."

- Classically: Searching an unstructured database of size N requires time $\Omega(N)$ — cannot do better than a linear scan.
- Quantum: Grover's algorithm is tight for unstructured search. Any quantum algorithm for the unstructured search problem requires making $\Omega(\sqrt{N})$ queries (to the function/database).

$\Rightarrow$ Quantum computes provide a quadratic speedup for unstructured search, and more broadly, function inversion.

Implications in cryptography: Consider a one-way function over a 128-bit domain. The task of inverting a one-way function is to find $x \in \{0,1\}^{128}$ such that $f(x) = y$ for some fixed target value f. Exhaustive search would take time $\approx 2^{128}$ on a classical computer, but using Grover's algorithm, can perform in time $\approx \sqrt{2^{128}} = 2^{64}$.

$\Rightarrow$ For symmetric cryptography, need to double key-sizes to maintain same level of security (unless there are new quantum attacks on the underlying construction itself.

$\Rightarrow$ Use AES-256 instead of AES-128 (not a significant change!)

Similar algorithm can be applied to obtain a quantum collision-finding algorithm that runs in time $\sqrt[3]{N}$ where N is the size of the domain (compare to $\sqrt{N}$ for the best classic algorithm)

↳ Instead of using SHA-256, use SHA-384 (not a significant change)

↳ The quantum algorithm require a large amount of space, so not clear that this is a significant threat, but even if it were, using hash functions with 384-bits of output suffices for security

Main takeaway: Symmetric cryptography mostly unaffected by quantum computers ~ generally just require a modest increase in key size

↳ e.g., symmetric encryption, MACs, authenticated encryption

Story more complicated for public-key primitives:

- Simon's algorithm and Shor's algorithm provide polynomial-time algorithms for solving discrete log (in any group with an efficiently-computable group operation) and for factoring
- Both algorithms rely on period finding (and more broadly, on solving the hidden subgroup problem)

Intuition for discrete log algorith (as a period finding problem):

- Let $(g, h = g^\alpha)$ be the discrete log instance in a group of prime order p
- Let $f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{G}$ be the function
$$f(x,y) = g^x h^{-y}$$
- By construction,
$$f(x+\alpha, y+1) = g^{x+\alpha} h^{-y-1} = g^x h^{-y} g^\alpha h^{-1} = g^x h^{-y} = f(x,y)$$
- Thus, the element $(\alpha, -1)$ is the period of f, so using Shor's algorithm, we can efficient compute $(\alpha, -1)$ from (g,h), which yields the discrete log of h

Thus, if large scale quantum computers come online, we will need new cryptographic assumptions for our public-key primitives
↳ All the algebraic assumptions we have considered so far (e.g., discrete log, factoring) are broken

How realistic is this threat? - Lots of progress in building quantum computers recently by both academia and industry (e.g., see initiatives by Google, IBM, etc.)

- To run Shor's algorithm to factor a 2048-bit RSA modulus, estimated to need a quantum computer with ≈ 2000 logical qubits (analog of a bit in classical computers)
  ↳ With quantum error correction, this requires millions of physical qubits to realize
  ↳ Today: machines with ~100 physical qubits, so still very far from being able to run Shor's algorithm
- Optimistic estimate: At least 10-15 years away

Should we be concerned? Quantum computers would break existing key-exchange and signature schemes

- Signatures: Future adversaries would be able to forge signatures under today's public keys, so if quantum computers come online, we can switch to and only use post-quantum schemes
- Key-Exchange: Future adversaries can break confidentiality of today's messages (i.e., we lose forward secrecy) — this is problematic in many scenarios (e.g., businesses want trade secrets to remain hidden for 50 years)

This course: will just focus on getting post-quantum signatures (will not discuss post-quantum key exchange)

[General approach for post-quantum cryptography: base hardness on assumptions believed to be hard on quantum computers (e.g., lattice-based cryptography, isogeny-based cryptography)]

For digital signatures, we can show that OWFs $\Rightarrow$ digital signatures
↳ Signatures can be based on symmetric primitives, so gives one approach to post-quantum signatures

For public-key cryptography, we will need new assumptions to get post-quantum security

We will see a brief flavor today — lattice assumptions

Learning with Errors (LWE): The LWE problem is defined with respect to lattice parameters n, m, q, χ, where χ is an error distribution over $\mathbb{Z}_q$ (oftentimes, this is a discrete Gaussian distribution over $\mathbb{Z}_q$). The $\mathsf{LWE}_{n,m,q,\chi}$ assumption states that for a random choice $A \xleftarrow{R} \mathbb{Z}_q^{n \times m}$, $s \xleftarrow{R} \mathbb{Z}_q^n$, $e \leftarrow \chi^m$, the following two distributions are computationally indistinguishable:

$$(A, s^T A + e^T) \overset{c}{\approx} (A, r)$$

where $r \xleftarrow{R} \mathbb{Z}_q^m$.

Symmetric encryption from LWE (for binary-valued messages) [Regev]

Setup(1^λ): Sample $s \xleftarrow{R} \mathbb{Z}_q^n$.

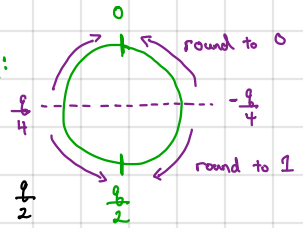
Encrypt$(s, \mu \in \{0,1\})$: Sample $a \xleftarrow{R} \mathbb{Z}_q^n$ and $e \leftarrow \chi$. Output $\left(a, s^T a + e + \mu \cdot \lfloor \frac{q}{2} \rfloor\right)$.

Decrypt(s, ct): Output $\underbrace{\lfloor ct_2 - s^T ct_1 \rceil_2}_{\text{"rounding operation"}}$

$$\lfloor x \rceil_2 = \begin{cases} 0 & \text{if } -\frac{q}{4} \le x < \frac{q}{4} \\ 1 & \text{otherwise} \end{cases}$$

↑ take $x \in \mathbb{Z}_q$ to be representative between $-\frac{q}{2}$ and $\frac{q}{2}$

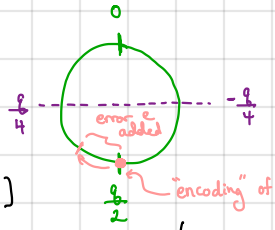
Visually:



Correctness: $ct_2 - s^T ct_1 = s^T a + e + \mu \cdot \lfloor \frac{q}{2} \rfloor - s^T a$

$= \mu \cdot \lfloor \frac{q}{2} \rfloor + e$

if $|e| < \frac{q}{4}$, then decryption recovers the correct bit



(message encrypted in "most significant bits" of the ciphertext)

↳ will see variant in HW5

Security: By the $\mathsf{LWE}_{n,m,q,\chi}$ assumption, $(a, s^T a + e) \overset{c}{\approx} (a, r)$ [$m = 1$]

where $r \xleftarrow{R} \mathbb{Z}_q$. Thus,

$$\left(a, s^T a + e + \mu \cdot \lfloor \tfrac{q}{2} \rfloor\right) \overset{c}{\approx} \left(a, r + \mu \cdot \lfloor \tfrac{q}{2} \rfloor\right)$$

↑ $r \xleftarrow{R} \mathbb{Z}_q$: one-time pad encryption of the message μ

Observe: this encryption scheme is additively homomorphic (over $\mathbb{Z}_2$):

$$\left(a_1, s^T a_1 + e_1 + \mu_1 \cdot \lfloor \tfrac{q}{2} \rfloor\right)$$
$$\left(a_2, s^T a_2 + e_2 + \mu_2 \cdot \lfloor \tfrac{q}{2} \rfloor\right)$$
$$\Rightarrow \left(a_1 + a_2,\ s^T (a_1 + a_2) + (e_1 + e_2) + (\mu_1 + \mu_2) \cdot \lfloor \tfrac{q}{2} \rfloor\right)$$

decryption then computes

$$(\mu_1 + \mu_2) \cdot \lfloor \tfrac{q}{2} \rfloor + e_1 + e_2$$

which when rounded yields $\mu_1 + \mu_2 \pmod 2$ provided that $|e_1 + e_2 + 1| < \frac{q}{4}$

This will give a simple approach for constructing a public-key encryption scheme from LWE