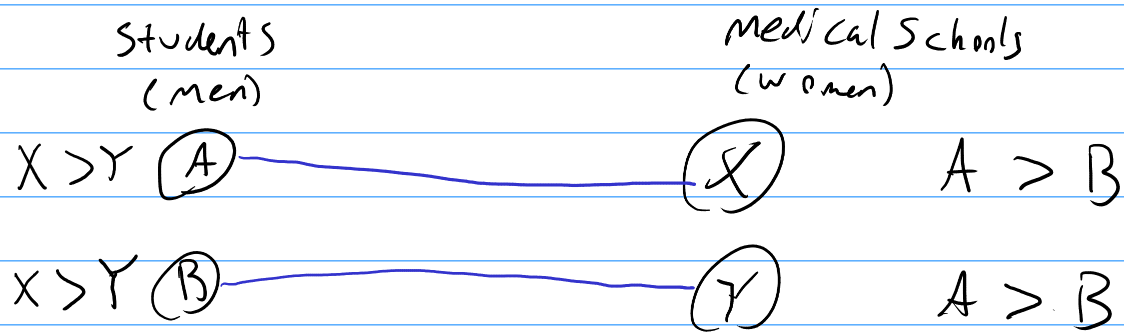
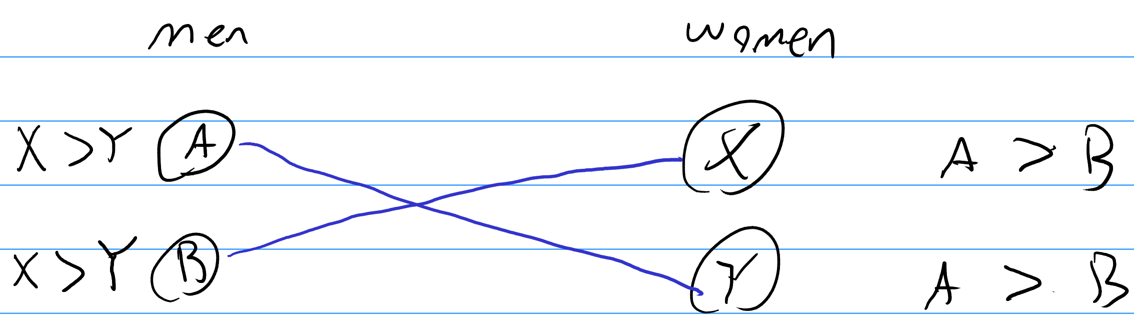
Stable Matching

(a.k.a. Stable Marriage)

Match n students
to n positions
each student/position has ranked preferences





unstable A & X will elope

Matching is stable if
no unmatched pair both prefer
each other to current partners.

Goal: given all preference lists, find a
stable matching.

Example

$Y > X > Z$ (A) — (X) $A > B > C$

$Y > X > Z$ (B) — (Y) $A > B > C$

$X > Y > Z$ (C) — (Z) $A > B > C$

Gale-Shapley:

Start with nobody matched.

Repeat:

Pick any unattached man.
This man proposes to next woman on his list.
The woman:
if unattached or preferred to current match, accepts & breaks up w/ current match
else, she prefers current match $\Rightarrow$ reject
if rejected, man crosses her off his list.

Claims

(1) This terminates.
- every woman's happiness never decreases.
- in each round, either:
  (1) name crossed off a man's list
  (2) woman gets happier by ≥ 1

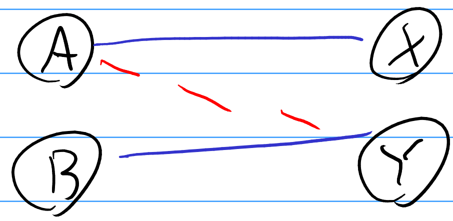
n men & n per list $\Rightarrow$ only n^2 times
n women & n per list $\rightarrow$ only n^2 times

$\Rightarrow O(n^2)$ steps

(2) The final state is stable

Suppose otherwise. Then in final state some A would elope with Y, while A matched to X & Y to B.

$Y > X$



$A > B$

How did this happen?

For A to be matched to X,

A proposed to X

$\Rightarrow$ A at some point proposed to Y.

$\Rightarrow$ at some point, Y had a mate she thought $\geq A$ in quality

But Y's happiness never decreases

$\Rightarrow$ Y's final happiness is $\geq$ her view of A.

which B is not, $\Rightarrow \Leftarrow$.

$\Rightarrow$ Gale-Shapley gives a stable matching in $O(n^2)$ time.

Theorem: Gale-Shapley is optimal for men & pessimal for women!

[of all stable matchings, A gets best match in any
X worst in any]

Lemma: each man only rejected by women that cannot be matched to him in any stable matching

Proof (induct on # rounds)

in a given round, suppose X rejects A in favor of B.

B has been rejected by all he prefers to X

$\Rightarrow$ all of those infeasible for B, by induction

$\Rightarrow$ in any stable matching, B would elope with X if X agrees.

$\Rightarrow$ in any stable matching, X is not matched to A (would cause elopement)

$\Rightarrow$ A infeasible for X.

$\Rightarrow$ Theorem.