Linear Programming Duality

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UT Austin

CS 331, Spring 2020 Coronavirus Edition

Plan for the class

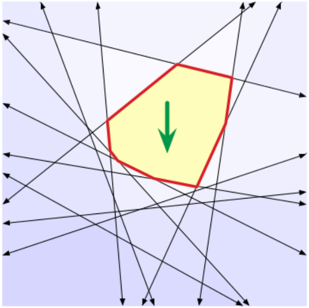
- Today: linear programming duality
- Tonight: problem set on LPs
- Last 2 weeks of class: complexity theory
- 1 problem set on complexity theory
- Final exam: given out after last class, due two days later.

Class Outline

1. LP Duality
2. Reducing Problems to Linear Programs

Linear Programming

- Maximize/minimize *linear objective* subject to *linear constraints*.



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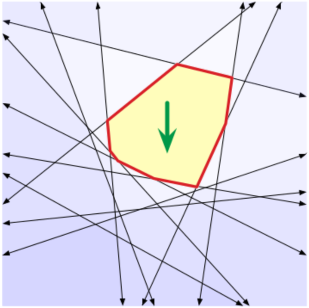
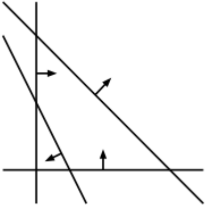


- Last class:
  - Solution lies at a *vertex* of *feasible region*.
  - Ways to translate between formulations ($\leq$ / $=$ / $\geq$, $x \geq 0$ or not)
  - Ways to solve (simplex)

Special Cases

- Infeasible: no possible answer.

$$
\begin{array}{ll}
\text{maximize} & x - y \\
\text{subject to} & 2x + y \leq 1 \\
 & x + y \geq 2 \\
 & x, y \geq 0
\end{array}
$$



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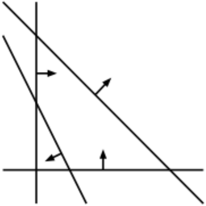
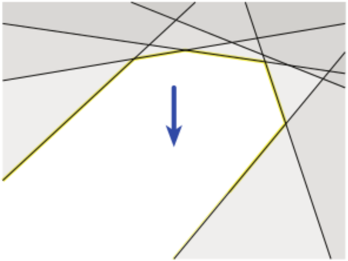


- Unbounded: infinitely good answer.



Linear Programming Upper bound?

- Cars & trucks example:

$$\begin{aligned} \text{maximize:} \quad & C + 2T && \text{(value)} \\ \text{subject to:} \quad & 2C + 3T \leq 12 && \text{(metal)} \\ & C + 5T \leq 15 && \text{(wood)} \\ & C, T \geq 0 \end{aligned}$$

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  - Is the answer larger than 20?

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  - No: $C + 2T \leq 2C + 3T \leq 12$.

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- Simple to prove a *lower bound* on answer OPT:
  - $OPT \geq 6$ because $(6, 0)$ possible.
- Question: can you easily show an *upper bound* on OPT?
  - Is the answer larger than 20?
  - No: $C + 2T \leq 2C + 3T \leq 12$.
  - But also:

$$C + 2T \leq C + \frac{8}{3}T = \frac{1}{3}((2C + 3T) + (C + 5T)) \leq \frac{1}{3}(12 + 15) = 9.$$

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- Get an *upper bound* by combining the constraints:

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- The above is $\frac{1}{3}$ of each. What is the best (α, β) combination?

$$\begin{aligned} \text{OPT} \le \quad & 12\alpha + 15\beta & \text{(value)} \\ \text{where:} \quad & 2\alpha + \beta \ge 1 & \text{(cars)} \\ & 3\alpha + 5\beta \ge 2 & \text{(trucks)} \\ & \alpha, \beta \ge 0 \end{aligned}$$

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Dual

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Primal			Dual	
maximize:	$c \cdot x$		minimize:	$b \cdot y$
subject to:	$Ax \leq b$	$\Longleftrightarrow$	subject to:	$A^T y \geq c$
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Primal solution $\leq$ Dual solution

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Primal solution $\leq$ Dual solution

- By construction, the dual is an upper bound on the primal.

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- Remarkable fact: the two are *equal*.

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- Remarkable fact: the two are *equal*.
  - This is "strong duality."
- Generalization of max flow-min cut theorem.

Linear Programming Duality: alternative forms

$$C + 2T \leq C + \frac{8}{3}T = \frac{1}{3}((2C + 3T) + (C + 5T)) \leq \frac{1}{3}(12 + 15) = 9.$$

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- Combine equations to get upper bound.
  - If C, T are negative, first step doesn't hold $\implies$ need equality.

$$\begin{array}{ll} & \text{“Alternative primal”} \\ \text{maximize:} & c \cdot x \\ \text{subject to:} & Ax \leq b \end{array} \implies \begin{array}{ll} & \text{“Alternative dual”} \\ \text{minimize:} & b \cdot y \\ \text{subject to:} & A^T y = c \\ & c \geq 0 \end{array}$$

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- Combine equations to get upper bound.
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  - If equations are equalities, can subtract them $\implies$ α, β can be < 0.

"Alternative primal"			"Alternative dual"	
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- Primal unbounded $\implies$ dual infeasible
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- Both infeasible is possible.
- Either one feasible and bounded $\implies$ other is too.

Linear programming duality

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Primal	Dual
Variables	$\Longrightarrow$ Constraints
Constraints	$\Longrightarrow$ Variables
Objective coefficients c	$\Longrightarrow$ Constraint values b
Constraint values	$\Longrightarrow$ Objective coefficients
Nonnegative vars	$\Longrightarrow$ Inequality constraints
Unconstrained vars	$\Longrightarrow$ Equality constraints
Unbounded	$\Longrightarrow$ Infeasible
Infeasible	$\Longrightarrow$ unbounded or infeasible
Nonzero variables	$\Longrightarrow$ tight constraints
Slack constraints	$\Longrightarrow$ zero variables

What do dual variables mean?

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- Solution: $(C, T) = (\frac{15}{7}, \frac{18}{7})$, $(\alpha, \beta) = (\frac{3}{7}, \frac{1}{7})$. Both give $\frac{51}{7}$.

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  - Check: 13 metal gives $(C, T) = (\frac{20}{7}, \frac{17}{7})$ for $\frac{54}{7} = \frac{51}{7} + \frac{3}{7}$.

What do dual variables mean? *Shadow prices!*

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- Dual variable α corresponds to the metal constraint.
- Tells you *marginal value of metal* to the factory:
  - With ϵ more metal, OPT will rise by $\alpha\epsilon$.
  - Check: 13 metal gives $(C, T) = (\frac{20}{7}, \frac{17}{7})$ for $\frac{54}{7} = \frac{51}{7} + \frac{3}{7}$.
- These are known as *shadow prices*.

Class Outline

1. LP Duality

2. Reducing Problems to Linear Programs

Writing old problems as linear programs

- Write network flow as a linear program
- Write shortest paths as a linear program
- Write minimum cut as a linear program

Maximum flow as a linear program

- Max flow is a linear program in the variables f_{uv} = flow from u to v:

$$\begin{aligned}
\text{maximize:} \quad & \sum_u f_{su} - f_{us} & & \text{(flow out)} \\
\text{subject to:} \quad & \sum_v f_{uv} - f_{vu} = 0 \quad \forall u \neq s, t & & \text{(conservation)} \\
& f_{uv} \leq C_{uv} \quad \forall u, v & & \text{(capacity)} \\
& f_{uv} \geq 0 & &
\end{aligned}$$

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& f_{uv} \leq C_{uv} \quad \forall u, v & & \text{(capacity)} \\
& f_{uv} \geq 0 & &
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- Computing the dual is a bit messy, but gives a min-cut formulation

Dual of the maximum flow LP

- Variables correspond to constraints: x_u for conservation constraints, y_{uv} for capacity (for all u, v).

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- A bit easier if we make the constraints include $u = s, t$.

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& f_{uv} \leq C_{uv} \quad \forall u, v & & \text{(capacity)} \\
& f_{uv} \geq 0
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  - This LP is special ("totally unimodular"): every vertex is integral.

Shortest paths as a linear program

- Stretching formulation:

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- Again, totally unimodular implies integral vertices.

Conclusion

- Every LP has a dual:
  - Motivated by certifying an upper bound.
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  - "Integer" LPs add a new constraint that $x \in \mathbb{Z}^n$. This is NP-hard.