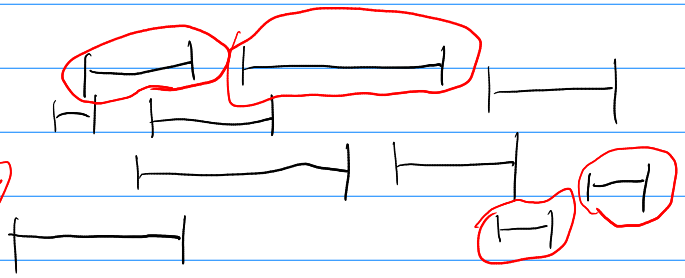
Interval Scheduling

Most basic version:

Given a set of n intervals:



Each defined by a start time s_i and finish time f_i

(think: people that want to rent your house).

What is the maximum number of disjoint intervals you can pick?

That is, find

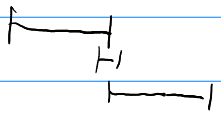
$S \subseteq [n]$ maximizing $|S|$
such that

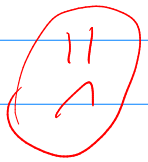
$$[s_i, f_i) \cap [s_j, f_j) = \emptyset \quad \forall i, j \in S, i \neq j$$

The example has 4.

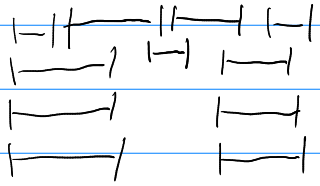
Many plausible greedy algorithms.

Shortest first?

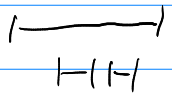




Interval w/ fewest overlaps first?



First come first serve?



Earliest finish time?

YES

but needs a proof to show it's different from the above fallacious arguments.

GreedySchedule (I): ← $I = \{(s_i, f_i)\}_{i \in [n]}$

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Sort I by increasing f_i.
S = []
t = -∞
for (s, f) in I:
    if s < t: continue
    t = f
    S.append((s, f))
return S.
```

Theorem. GreedySchedule returns a valid set of intervals of maximum size.

Proof Because $s \leq f$ for every interval, t never decreases. Therefore t is always the largest f in S. This ensures each new interval added to S does not overlap with any interval already in S, so the returned S is valid.

We now prove by induction on n that the returned S has maximal size. The base case of $n = 0$ is trivial. Now consider some $n \geq 1$, and assume the theorem holds for all I' with $|I'| \leq n-1$.

For any time t, define $I_t = \{(s,f) \in I \mid s \geq t\}$. Notice that, if (s_1, f_1) has the earliest finish time in I, then

$$\text{GreedySchedule}(I) = [(s_1, f_1)] + \text{GreedySchedule}(I_{f_1}).$$

Let S^* be a maximum disjoint subset of I_f which we can write out in order as

$$S^* = (s_1^*, f_1^*), (s_2^*, f_2^*), \ldots, (s_{k^*}^*, f_{k^*}^*)$$

for $k^* = |S^*|$. Similarly, GreedySchedule returns an S with

$$S = (s_1, f_1), (s_2, f_2), \ldots, (s_k, f_k)$$

for $k = |S|$. We claim $k = k^*$.

Since f_1 is the earliest possible finish time,

$$f_1 \leq f_1^* \leq s_2^*.$$

Therefore $(s_2^*, f_2^*), \ldots, (s_{k^*}^*, f_{k^*}^*)$ all lie in I_{f_1}. Since these are disjoint, the optimal solution for I_{f_1} has at least k^*-1 intervals. By the inductive hypothesis, this means

$$|\text{GreedySchedule}(I_{f_1})| \geq k^* - 1$$

so $k = |\text{GreedySchedule}(I)| \geq k^*$.

Since k^* is the maximum possible, $k^* = k$ as desired.

So: $O(n \log n)$ sort $+ O(n)$ time.

Weighted interval scheduling

Each interval now has a weight w_i.

Maximize $\sum_{i \in S} w_i$

over sets of disjoint intervals.

Naive recursion:

Sched (I)

Let i = first elt of I

Return min of

not chosen: Sched $(I \setminus i)$,

chosen: Sched $(I \setminus \{i$ or anything conflicting with $i\})$ + w_i.

Problem: 2^n possible inputs.

Solution: If I sorted by f_i, then only $n+1$ inputs ever happen:
(suffix of I sorted by s_i)

$\Rightarrow$ memoized time is $n \cdot$ (time per input)

n^2 naively

n more carefully.

Sched (index in I, f of last chosen)