Linear Programming

Developed in WW2 era, Leonid Kantorovich

n variables, m constraints
optimize linear function
subject to linear constraints.

e.g. $5x_1 + 6x_2 - 3x_3 + x_4$

s.t. $x_3 \geq x_1 + 2x_2 + 3$

$x_4 = x_1 + x_2$

$x_1 \geq 0$

$x_2 \leq 1$

"standard form": or "symmetric"

max $c^T x$

s.t. $x \geq 0$

$Ax \leq b$

alternative form: no $x \geq 0$ requirement.

max $c^T x$

s.t. $Ax \leq b$.

$x = y - z$, $y, z \geq 0$

max $c^T (y - z)$

s.t. $A(y-z) \leq b$

$y, z \geq 0$.

alternative form:

$x_1 + 2x_2 - x_3 \leq -3$

$x_1 + x_2 - x_4 \leq 0$

$-x_1 - x_2 + x_4 \leq 0$

$-x_1 \leq 0$

$x_2 \leq 1$

equational form:

max $c^T x$

s.t. $Ax = b$, $x \geq 0$

slack variables: $z = Ax - b$

max $c^T x$

$Ax + z = b$

$x, z \geq 0$

$[A \mid I]\begin{bmatrix} x \\ z \end{bmatrix} = b$

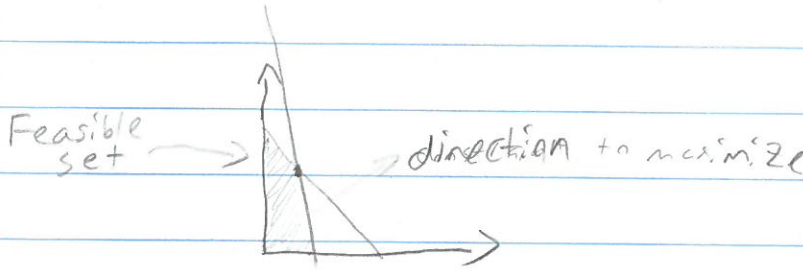
example:

$$\max \quad 6x + 3y$$

$$\text{s.t.} \quad x \geq 0, \; y \geq 0$$

$$2x + 2y \leq 5$$

$$5x + y \leq 4$$



will be maximized at a vertex.

But many possible vertices

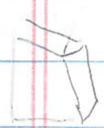
n unknowns, so n equations give unique solution (typically)

$\rightarrow \binom{m+n}{n} \leq (m+n)^n$ vertices.

Simplex method: walk along vertices

Given a vertex, walk edges

edge corresponds to an equation to relax.



Move along steepest improvement direction, till another constraint tight. ($\Rightarrow$ hit another vertex)

Interior point (ellipsoid): polynomial. $n^4 \cdot L$ L bits precision

In practice: depends on problem.

Open: strongly polynomial?

LP $\iff$ "does this ~~polytope~~ system have any solutions"

binary search on $\beta \geq c^T x$, add $c^T x \geq \beta$ as constraints.

Of course, max flow is an LP.

$$\max \sum f_{su} \qquad c^T x$$

s.t. $\sum_v f_{uv} - f_{vu} = 0 \ \forall u \neq s,t \qquad Ax \leq b$

$f_{uv} \leq c_{uv} \quad \forall u,v.$

$f_{uv} \geq 0$

min cut? equivalent minimization problem.

$$\min \sum Y_{uv} c_{uv} \qquad Y_{uv},\ Y_u. \quad b^T y$$

s.t. $\qquad A^T y \geq c$

(1) $Y_{uv} - Y_u + Y_v \geq 0 \qquad (u,v),\ u,v \neq s$

$Y_{sv} + Y_v \geq 1 \qquad (s,u)$

$Y_{ut} - Y_u \geq 0 \qquad (u,t)$

$Y_{ut} \geq 0$

Y_{uv} = "is u-v cut: $u \in S$, $v \notin S$" 1/0

Y_u = "is $u \in S$? 1 if true, 0 otherwise"

$Y_u, Y_v, Y_{uv} \in \{0,1\}$

(1) $u \in S \Rightarrow v \in S$ or uv cut.

(3) $u \in S \Rightarrow ut$ cut $\qquad$ same, but $Y_s = 1$

(2) $v \notin S \Rightarrow sv$ cut. $\qquad$ same, say $Y_t = 0$

not obviously integral.

Duality:

Primal $P = \max \; 6x+3y$

s.t. $x \geq 0, \; y \geq 0$

$2x+2y \leq 5$

$5x+y \leq 4$

$x+3y \leq 3$

Any upper bound?

$6x+3y \leq 6x+6y \leq 3 \cdot 5 = 15.$

$6x+3y \leq 7x+3y \leq 5+4 = 9$

Let α, β satisfy

$2\alpha + 5\beta + \gamma \geq 6 \qquad \alpha, \beta \geq 0$

$2\alpha + \beta + 3\gamma \geq 3$

then $6x+3y \leq 5\alpha + 4\beta + 3\gamma$

So $D = \min \; 5\alpha + 4\beta + 3\gamma$

s.t. $2\alpha + 5\beta + \gamma \geq 6$ — another LP

$2\alpha + \beta + 3\gamma \geq 3.$

$\alpha, \beta, \gamma \geq 0$

$P \leq D$ ← weak duality (easily shown)

$P = D$ ← strong duality (harder. True for LP)

Canonical standard

$\max \; c^T x \iff \min \; b^T y$

s.t. $Ax \leq b$ s.t. $A^T y \geq c$

$x \geq 0$ $y \geq 0$

$\max \; c^T x$, $Ax \leq b$ ⇔ $\min \; b^T y$ s.t. $A^T y = c$

non-negative variables $\Leftrightarrow$ inequality constraints
unconstrained variables $\Leftrightarrow$ equality constraints

x_1 = # flat-bed produced / month
x_2 = # economy
x_3 = # luxury

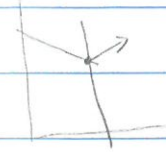
$\frac{1}{2}x_1 + 2x_2 + x_3 \leq 24$ days of metalworking
$x_1 + 2x_2 + 4x_3 \leq 60$ days of wood working.
$x_1, x_2, x_3 \geq 0$

Max $z = 6x_1 + 14x_2 + 13x_3$

Dual: minimize $24y_1 + 60y_2$

s.t. $y_1, y_2 \geq 0$

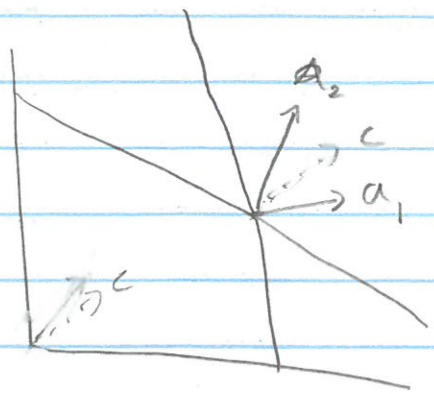
$\frac{1}{2}y_1 + y_2 \geq 6$
$2y_1 + 2y_2 \geq 14$
$y_1 + 4y_3 \geq 13$



3 options,

$y_1 = 11, \quad y_2 = \frac{1}{2} \rightarrow 294$

$x_1 = 36, \quad x_3 = 6 \rightarrow 216 + 78 = 294$
Build 36 flat-beds, 6 luxury
One more day of metalworking is worth 11
wood $\frac{1}{2}$



not nonnegative

P: $\max \; c^T x \;\; | \;\; Ax \le b$

D: $\min \; b^T y \;\; | \;\; A^T y = c, \;\; y \geq 0$

P: finds x vertex, $x + \lambda c$ infeasible $\forall \lambda > 0$

<u>Proof of Duality</u>

We know: $P \le D$, weak duality

But also: $P = D$

Let $x^* =$ opt for P, I be tight constraints.

Then c lies in cone at x^*: $a_i^T x^* = b_i$

$c = A^T y \quad y \geq 0$

$\mathrm{supp}(y) \subseteq I$

$$b^T y = \sum_{i \in I} b_i y_i = \sum_{i \in I} (a_i^T x^*) y_i = \sum_{i \in I} a_i y_i x^*$$

$$= c^T x^*$$

$$= \sum_i \sum_j (a_{ij} x_j^*) y_i = \sum_j x_j^* \sum_i a_{ij} y_i$$