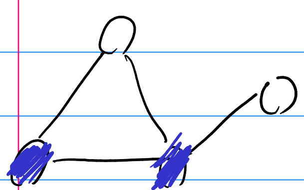
Approximation Algorithms

What can we do with NP-hard problems?
Can't find optimal solution.
idea: <u>approximately</u> optimal solutions

Vertex Cover: $G = (V, E)$
Find $S \subseteq V$ s.t. $e \cap S \neq \emptyset \;\; \forall e \in E$
of minimum size $|S|$.



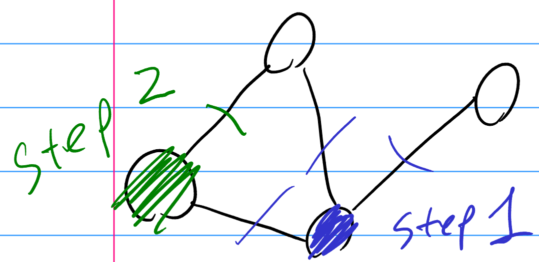
Shown NP-hard last week.

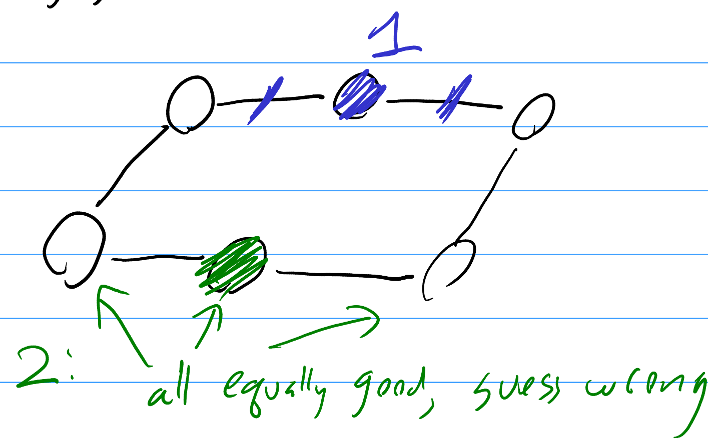
If best answer is OPT,
Can we find vertex cover S w/ $|S| \leq \alpha \cdot OPT$
for some "approximation factor" α?

What is smallest $\alpha > 1$ s.t. this is possible in polynomial time?

Idea: <u>greedy</u>.

- repeatedly add vertex that covers most additional edges.





get 4 not 3. $\Rightarrow \alpha \geq 4/3$

Can we prove upper bound on α?

G_i = graph left in i^{th} step $(G_1 = G)$ S

d_i = # additional edges covered in i^{th} step

= degree of i^{th} node picked in G_i

S^* = optimal cover

$$\sum_{u \in S^*} \deg(u) \geq m \Rightarrow \underset{u \in S^*}{\text{average}} \deg(u) \geq \frac{m}{OPT}$$

$$\Rightarrow \quad d_1 \geq \frac{m}{OPT}$$

$$d_i \geq \frac{|E(G_i)|}{OPT(G_i)} \geq \frac{m - \sum_{j<i} d_j}{OPT}$$

How long till $|E(G_i)| \leq \frac{m}{2}$?

- as long as untrue, decreases by $\frac{m}{2 \cdot OPT}$

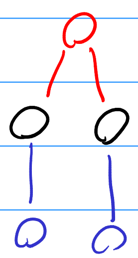
$\Rightarrow$ $\leq OPT$ steps.

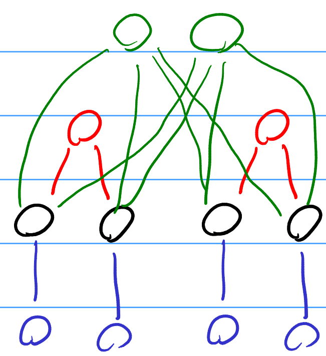
$\Rightarrow$ $OPT \cdot \log m$ steps till G_i empty

$\Rightarrow$ $|S| \leq OPT \cdot \log m = OPT \cdot O(\log n)$

$\Rightarrow$ $\alpha = O(\log n)$

This is tight:

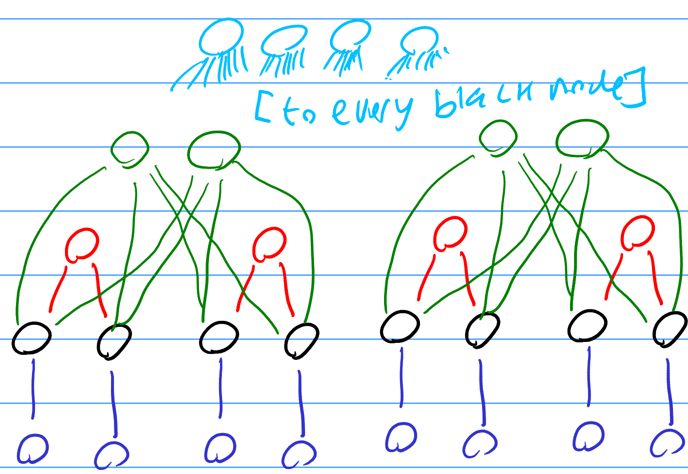




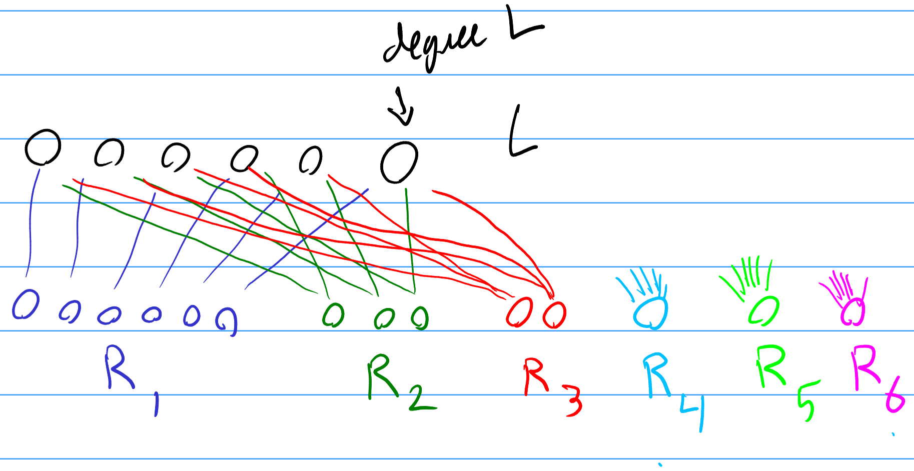
Greedy can pick  in order

$\alpha = 3/2$

$\alpha = \frac{8}{4} = 2$



$\alpha = \frac{20}{8} = 2.5$



$|R_i| = \frac{|L|}{i}$

OPT = pick top L.

Greedy could take R_6, R_5, R_4, R_3, R_2, R in order $\Rightarrow$ $L \cdot \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{L}\right) \approx L \log L$ nodes

$\Rightarrow \alpha = \Theta(\log L) = \Theta(\log n)$

Can we do better?

Stupid Vertex Cover(G):

Repeatedly pick $\underline{\text{any}}$ $e = uv \in E$

add both u & v to S

remove all edges touching u or v.

Claim: Stupid Vertex Cover is 2-approximation.

(1) result is vertex cover. any edges removed are covered.

(2) result is small:

$S^* = OPT$,

$\tilde{E}$ = set of edges picked in Stupid Vertex Cover.

each $e \in \tilde{E}$ intersects S^* ≥ 1 time,

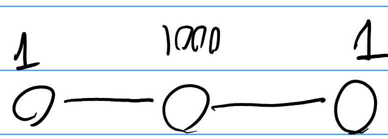
& no two $e, e' \in \tilde{E}$ overlap

$\Rightarrow |\tilde{E}| \leq |S^*|$

$\Rightarrow |S| = 2|\tilde{E}| \leq 2|S^*|$.

Q: $\underline{\text{weighted}}$ vertex cover?

$\min \sum_{u \in S} w_u$ not $|S|$



Greedy: 1000

Stupid: 1001

OPT: 2

Technique: Linear Programming Relaxations

① Write as Integer Linear Program:

$$\min \sum w_u x_u$$

$$x_u \in \{0, 1\}$$

$$x_u + x_v \geq 1 \quad \forall (u, v) \in E$$

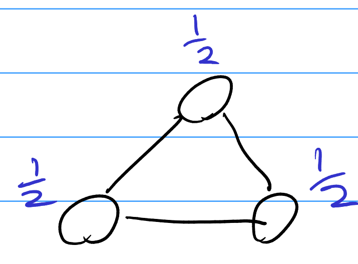
② Relax to Linear Program:

$$\min \sum w_u x_u$$

$$x_u + x_v \geq 1 \quad \forall (u, v) \in E$$
$$0 \leq x_u \leq 1$$

③ Solve, getting optimal <u>fractional</u> solution

$$\text{FRACOPT} \leq \text{OPT}$$



④ <u>Round</u> to integer solution not much worse:

FRACOPT = 3/2
OPT = 2

$$\text{Value} \leq \alpha \cdot \text{FRACOPT}$$

$$\Rightarrow \quad \leq \alpha \cdot \text{OPT} \quad \Rightarrow \alpha\text{-approximation}$$

Rounding: get fractional solution X

$$\bar{X}_u = \begin{cases} 1 & \text{if } X_u \geq \frac{1}{2}, \\ 0 & \text{otherwise} \end{cases}$$

$$X_u + X_v \geq 1 \Rightarrow \bar{X}_u + \bar{X}_v \geq 1$$

So $\bar{X}$ is valid cover.

$$\text{Value}(\bar{X}) = \sum w_u \bar{x}_u \leq \sum w_u x_u \cdot 2$$

$$= 2 \cdot \text{FRACOPT}$$

$$\leq 2 \cdot \text{OPT}$$

$\Rightarrow$ 2-approximation.

Another Technique: Randomized Rounding

$$\bar{x}_u = \begin{cases} 1 & \text{with probability } \alpha \cdot x_u \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow E[\text{value}(\bar{x})] = \alpha \cdot \text{FRAC OPT} \leq \alpha \cdot \text{OPT}$

<u>expected</u> α-approximation

But is it a vertex cover?

Prob. edge (u, v) uncovered

$$= (1 - \alpha x_u)(1 - \alpha x_v)$$

$$\leq e^{-\alpha(x_u + x_v)} \leq e^{-\alpha}$$

$e^{-y} \geq 1 - y$

So set $\alpha = \log n$ to be VC with high probability.

Not so useful for vertex cover, but:

lightest <u>set</u> cover, where "vertex" instead covers an arbitrary set of items:

$$\min \sum w_u x_u$$

$$x_u \in \{0, 1\}$$

$$\sum_{u \ni e} x_u \geq 1 \quad \forall e \in E$$

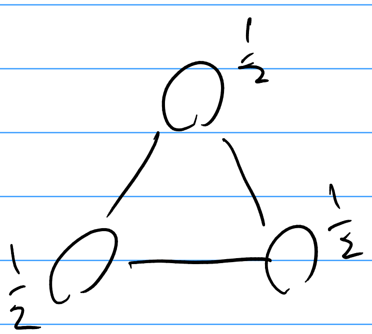
the same approach works, and Stupid Vertex Cover does not.

Q: Can we find better rounding scheme to get $\alpha < \frac{1}{2}$?

A: <u>No</u>, because "Integrality Gap" is 2.

$$\text{Integrality gap} = \max_{\text{instances}} \frac{OPT}{FRACOPT}$$

e.g.:



$FRACOPT = 3/2$
$OPT = 2$

$\Rightarrow$ Integrality gap $\geq 4/3$.

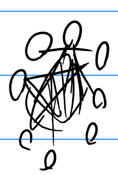
If we use outline:

$$\text{Value}(\text{round}(x)) \underset{①}{\leq} \alpha \cdot \text{Value}(x) = \alpha \cdot FRACOPT \leq \alpha \cdot OPT$$

then step ① <u>must</u> lose $\alpha \geq \frac{4}{3}$ on triangle

$$\text{Value}(\text{round}(x)) \geq OPT = 2$$
$$\text{Value}(x) = FRACOPT = 3/2.$$

Clique:



$FRACOPT = \frac{n}{2}$
$OPT = n-1$

$\Rightarrow$ gap $= 2\left(1 - \frac{1}{n}\right) = 2 - o(1)$

$\Rightarrow$ outline requires $\alpha = 2$.

Oddity: The instance w/ the integrality gap is <u>easy</u>

- if $OPT = 2(1-\frac{1}{n})$ FRACOPT

then $\text{value}(\text{rand}(x)) \le 2 \cdot \text{FRACOPT}$

$$= \underbrace{(1+\tfrac{1}{n-1})}_{\text{very good!}} \cdot OPT$$

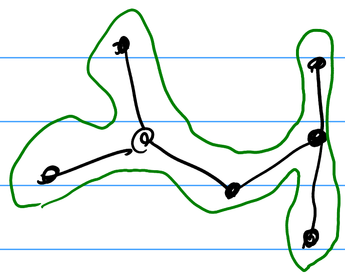
but still <u>suggests</u> hardness; would need a different outline [approximation factor different by instance] that is rare to get working.

Here: $2-o(1)$ is best known alg.

Another problem: traveling salesman (TSP)

metric TSP: distances satisfy triangle inequality $d(a,b) \le d(a,c) + d(c,b)$

Find cycle visiting all of min total length.

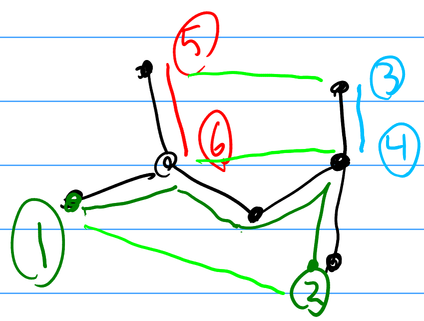


Idea 1: Min Spanning Tree.

Claim: OPT $\ge$ MST

Gives route: Follow around MST edge.
$2 \cdot \text{MST} \le 2 \cdot \text{OPT} \Rightarrow$ 2-approx

Christofides:



Split MST into paths
vertex is path endpoint
$\Leftrightarrow$ degree in MST is odd

Claim: given any matching on odd degree vertices, get TSP tour.
"minimum odd-degree matching" =: MOM

True Tour passes through odd-degree vertices:

$$V_1 \rightsquigarrow V_2 \rightsquigarrow V_3 \rightsquigarrow V_4 \rightsquigarrow \cdots \rightsquigarrow V_{2k}$$

giving 2 odd-degree matchings, so $2 \cdot \text{MOM} \le \text{OPT}$
$\Rightarrow \text{MST} + \text{MOM} \le \frac{3}{2}\,\text{OPT}$