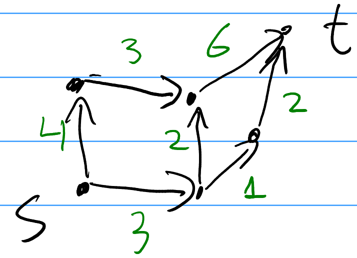
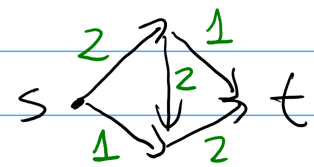
Network flow & minimum cut

Two graph problems, tightly connected

Given a weighted directed graph w/ nodes s, t.



weight $c(e) \geq 0 \quad \forall e \in E$



Minimum s-t cut

$c(e)$ = cost to remove edge e

Q: Cost to make t unreachable from s?

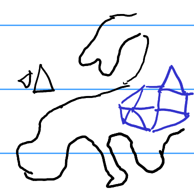
"cut" $S \subseteq V$

"s-t cut": cut S s.t. $s \in S$, $t \notin S$

$$\text{cost}(S) = \sum_{\substack{(u,v) \in E \\ u \in S, v \notin S}} c(u \to v)$$

Q: what is min cost(S) over s-t cuts S?

early example:



Graph = Soviet rail lines
s = Moscow
t = East Germany
$c(u \to v)$ = cost of bombing rail line

US Air Force planning which Soviet rail lines to bomb to stop supplies reaching East Germany

Maximum s-t flow

$c(e)$ = capacity to move stuff (supplies, water) along e

Q: how much can be sent $s \rightsquigarrow t$?

Flows

function f: $f(u \to v) \geq 0 \quad \forall\, u, v \in V$

$$\partial f(u) := \text{net flow out of } u$$
$$= \sum_{v} f(u \to v) - f(v \to u)$$

"s-t flow" is f satisfying

Conservation Constraint:

$$\partial f(u) = 0 \quad \forall\, u \neq s, t$$

"feasible" s-t flow f satisfies

Capacity Constraint:

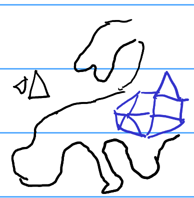
$$f(u \to v) \leq c(u \to v) \quad \forall\, u, v$$

"Value" of flow: $|f| := \partial f(s)$
$$= \sum_{v} f(s \to u) - f(u \to s)$$

Maximum Flow Question:

max $|f|$ over feasible s-t flows

Example:



Graph = Soviet rail lines

s = Moscow

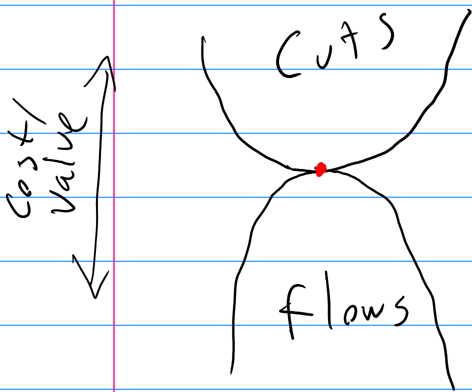
t = East Germany

$c(u \to v)$ = # supplies rail line can send / day

Same one, but Soviet view: how fast can we supply front?

Main Theorem: Max flow = min cut

("max-flow min-cut" theorem)



Two parts:

(1) any cut $\geq$ any flow

$\forall f, S$: $\quad Cost(S) \geq |f|$

(2) some cut = some flow

$\exists f, S$: $\quad Cost(S) \leq |f|$

proof by algorithm

(1) Take any f, S. $\qquad s \in S,\ t \notin S$.

$$|f| = \partial f(s) = \sum_{v} f(s \to v) - f(v \to s)$$

Conservation

$$= \sum_{u \in S} \partial f(u) = \sum_{\substack{u \in S \\ v \in V}} f(u \to v) - f(v \to u)$$

if $v \in S$, edge appears twice & cancels, so

$$|f| = \sum_{\substack{u \in S \\ v \notin S}} f(u \to v) - f(v \to u)$$

Capacity

$$\leq \sum_{u \in S, v \notin S} c(u \to v) - 0 = Cost(S)$$

Note: equality $\iff$ $S \to \bar{S}$ "saturated": $f(u \to v) = c(u \to v)$

$\bar{S} \to S$ empty $f(u \to v) = 0$

(2) Proof approach. (Ford-Fulkerson)

Given any flow f

find "augmenting path" in "residual graph"

gives new flow f' w/ $|f'| > |f|$

unless no path exists

$\Rightarrow$ reachable nodes form cut S w/ $|f| = \text{Cost}(S)$

$\Rightarrow$ $\exists f, S$ with $|f| = \text{cost}(S)$

More Notation

Given flow f: $0 \le f(u \to v) \le c(u \to v)$

"net flow" $f_{net}(u \to v) := f(u \to v) - f(v \to u)$

anti symmetry $\quad f_{net}(u \to v) = -f_{net}(v \to u)$

Capacity $\quad f_{net}(u \to v) \le c(u \to v) \quad \forall u, v$

Conservation $\quad \partial f(u) = \sum_v f_{net}(u \to v)$

has $\partial f(u) = 0 \quad \forall u \neq s, t$

Can go from regular flow to net flow & back

back: $f(u \to v) = \max(0, f_{net}(u \to v))$

Residual graph

Given flows f & capacities c,

$c_f(u \to v) := c(u \to v) - f_{net}(u \to v)$

"how much more you can send $u \to v$"

You've used up some, giving less capacity

OR: some flow is coming the other way you can cancel

Augmenting Path

Path in residual graph over edges w/ $c_f(u \to v) > 0$

Lemma If no $s \to t$ augmenting path exists in residual graph, then $S :=$ nodes reachable from s is an s-t cut w/ $\text{cost}(S) = |f|$

pf

$$\forall u \in S, v \notin S, \quad c_f(u \to v) = 0$$
$$\Rightarrow \quad f_{net}(u \to v) = c(u \to v)$$

$$\text{cost}(S) = \sum_{u \in S, v \notin S} c(u \to v)$$
$$= \sum_{u \in S, v \notin S} f_{net}(u \to v)$$

Now, $\sum_{u,v \in S} f_{net}(u \to v) = 0$ by antisymmetry

$$\Rightarrow \quad \text{cost}(S) = \sum_{u \in S} \sum_{v \in V} f_{net}(u \to v)$$
$$= \sum_{u \in S} \partial f(u) = \partial f(s) = |f| \quad \blacksquare$$

So: no augmenting path $\Rightarrow$ done, flow = cut

Finally: $\exists$ augmenting path $\Rightarrow$ can increase f

Suppose $\exists$ augmenting path

$P: \quad s = u_0 \to u_1 \to \ldots \to u_k = t$

$$\Delta = \min_i c_f(u_i \to u_{i+1}) > 0$$

New flow f':

$$f'_{net}(u \to v) = f_{net}(u \to v) + \begin{cases} \Delta & \text{if } u \to v \in P \\ -\Delta & \text{if } v \to u \in P \\ 0 & \text{o/w} \end{cases}$$

Lemma: f' is feasible s-t flow

antisymmetry: ✓

Conservation:

$\partial f(u)$ unchanged for $u \neq s, t$:

Δ comes in & leaves. ✓

Capacity:

$u \to v \notin P \Rightarrow f'_{net}(u \to v) \leq f_{net}(u \to v) \leq c(u \to v)$

$u \to v \in P \Rightarrow$

$$\begin{aligned} f'_{net}(u \to v) &= f_{net}(u \to v) + \Delta \\ &\leq f_{net}(u \to v) + c_f(u \to v) \\ &= c(u \to v) \end{aligned}$$ ✓

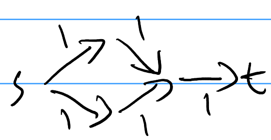
And $\partial f'(s) = \partial f(s) + \Delta > \partial f(s)$

Theorem if capacities are integers
max flow is F
then $\leq F$ flows found
$\Rightarrow O(mF)$ time algorithm for max flow/min cut

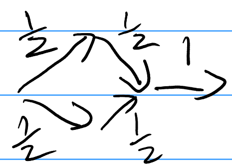
(How to find cut? nodes reachable by unsaturated edges)

Note: implies if $c(u \to v)$ integer,
$\exists$ max flow w/ all flows integer

$c(u \to v)$



$f(u \to v)$



Running Times

not Polynomial $O(mF)$ any path
large numbers can make slow

unit weights: $O(mn)$

weakly Polynomial $O((m+n\log n)\cdot m\log F)$

<u>Largest</u> augmenting path
Dijkstra w/ (max, min)
(= Prim's, negated)

strongly Polynomial $O(m^2n)$ <u>Shortest</u> augmenting path
(= BFS)

$O(mn^2)$ blocking flows

$O(mn\log n)$ fancy

Edmonds-Karp:

Ford-Fulkerson w/ BFS

$O(m^2 n)$ time (not $O(mF)$)

Idea: show distances from source increases over time, as shortest paths get removed

(Problem: sometimes no distance increases)

For flow f, source s, arbitrary $v \in V$

Let $\delta_f(v)$ = shortest $s \rightsquigarrow v$ path in residual graph G_f (i.e. augmenting path)

Claim: $\forall v$, $\delta_f(v)$ is nondecreasing over time as f evolves.

Pf Suppose not. Then $\exists f, v,$ and augmenting path P so augmented flow f' has

$$\delta_{f'}(v) < \delta_f(v) \qquad (\star)$$

Let v minimize $\delta_{f'}(v)$ with this property.

Let shortest path in $G_{f'}$ be

$$s \rightsquigarrow u \rightarrow v$$

Then:

$$\delta_f(u) \le \delta_{f'}(u) = \delta_{f'}(v) - 1$$

Now, either:

(A) $v \to u$ edge on augmenting path P.

$$\Rightarrow \delta_f(v) = \delta_f(u) - 1$$ because P shortest path

(B) $v \to u$ edge not on augmenting path P

$\Rightarrow$ $u \to v$ edge already was in G_f.

$$\Rightarrow \delta_f(v) \le \delta_f(u) + 1$$

Either way:

$$\delta_f(v) \le \delta_f(u) + 1 \le \delta_{f'}(v)$$

$\Rightarrow \not\leq$ (by ✡) ∎

So, distances don't decrease. Now: they sometimes increase.

- Augmenting flow removes saturated edges
- ≥ 1 edge is saturated.

When $u \to v$ is saturated: $u \to v$ lies on P, so

$$\delta_f(v) = \delta_f(u) + 1$$

& $u \to v$ removed from residual graph.

For $u \to v$ to be added back to residual graph, need to augment along $v \to u$

$\Rightarrow$ later f' s.t.

$$\delta_{f'}(u) = \delta_{f'}(v) + 1$$
$$\geq \delta_f(v) + 1 = \delta_f(u) + 2$$

So: every time $(u \to v)$ returns to graph, $\delta(u)$ is ≥ 2 more than last time

$0 \leq \delta_f(u) \leq n-1$, so

every edge saturated $\leq \frac{n+1}{2}$ times.

$\leq 2m$ edges $\Rightarrow$ $O(mn)$ total saturations

$u \to v$ and $v \to u$

$\Rightarrow$ $O(mn)$ rounds

$\Rightarrow$ $O(m^2 n)$ time. ∎