CS 388R: Randomized Algorithms Fall 2017

Lecture 13 — October 12, 2017

Prof. Eric Price *Scribe: Eirini Asteri*

NOTE: THESE NOTES HAVE NOT BEEN EDITED OR CHECKED FOR CORRECTNESS

1 Overview

In the last lecture,

- we explored the problem of finding matchings in bipartite graphs,
- we introduced the problem of on-line bipartite matching,
- we observed that any deterministic algorithm for on-line bipartite matching produces an 1/2-approximation and,
- we studied a simple randomized algorithm with approximation ratio $(1/2 + O(1/\sqrt{n}))$.

In this lecture,

- we will study an $(1 - 1/e)$-approx. randomized algorithm for the problem of on-line bipartite matching given by Karp, Vazirani and Vazirani [2].

2 Online Bipartite Matching

2.1 Introduction

Let $G(U, V, E)$ be a bipartite graph on $2n$ vertices such that G contains a perfect matching. We can think of the situation where we have n customers (U) and n merchants(V). We assume that the customer vertices arrive in a preselected order, and that the edges incident to a vertex are revealed to us only when the vertex (customer) arrives. The task is to decide, as each customer arrives, which eligible merchant to assign to him, so that the size of the matching obtained is maximized.

2.2 The Ranking Algorithm

In [2] the authors introduced the Ranking algorithm for the Online Bipartite Matching. The algorithm achieves an expected competitive ratio of $1 - 1/e$.

Ranking Algorithm

– **Initialization** Pick a random permutation of the merchant vertices, thereby assigning to each merchant a random priority or ranking.

– **Matching Phase** As each customer arrives, match him to the eligible merchant (if any) of highest rank.

After initialization, there is an ordering on both the customer and merchant vertices (the preselected arrival ordering on customers and the randomly chosen ordering on merchants).

Notation

- Given an arrival order π for U and a rank order σ for V, the ranking algorithm outputs Matching(G, π, σ).

- $M(x) :=$ vertex matched to vertex x in the perfect matching

Lemma 1. *Let $H = G - \{x\}$ and π_H, σ_H permutations on H. Then, Matching$(H, \pi_H, \sigma_H) =$ Matching$(G, \pi_G, \sigma_G) + ($a single augmenting path$)$.*

Lemma 2. *Let $u \in U$ and $v = M(u)$. If v is not matched at all in Matching(G, π, σ) , then u is matched to some v' with $\sigma(v') < \sigma(v)$.*

Lemma 3. *Let x_t be the probability that a t-rank vertex is matched, then*

$$1 - x_t \leq \frac{1}{n} \sum_{s \leq t} x_s.$$

Proof. (Intuitive, but incorrect)

Let v be the rank t vertex under σ. Since σ is uniformly random, v and $u = M(v)$ are also uniformly random. Let $R_{t-1} \subseteq U$ be the subset of vertices that are matched to vertices on the right with rank at most $t - 1$ under Matching(G, π, σ). Then,

$$\mathbb{E}[|R_{t-1}|] = \sum_{s \leq t-1} x_s.$$

The vertex v of rank t is unmatched if and only if $u \in R_{t-1}$.

$$\mathbb{P}[v \text{ is unmatched}] = \mathbb{P}[u \in R_{t-1}] = \mathbb{E}_{R_{t-1}}[\mathbb{P}[u \in R_{t-1}]] \leq \mathbb{E}_{R_{t-1}} \left[\frac{|R_{t-1}|}{n} \right] = \frac{\sum_{s \leq t-1} x_s}{n}$$

However, this proof is incorrect since u and R_{t-1} are not independent. They both depend on the random permutation. $\square$

Lemma 4. *Let $u \in U$ and $v = M(u)$. Let σ be some permutation and $\sigma^{(i)}$ be the permutation obtained by shifting v to rank i. If v is not matched in σ, then u is matched under $\sigma^{(i)}$ to some $v^{(i)}$ with $\sigma^{(i)}(v^{(i)}) \leq \sigma(v)$.*

Proof. Let u be matched to v' under σ, and to $v^{(i)}$ under $\sigma^{(i)}$, then

$$\sigma^{(i)}(v^{(i)}) \leq \sigma^{(i)}(v') \leq \sigma(v') + 1 \leq \sigma(u)$$

□

Proof. (Correct proof of Lemma 3) Choose a random σ and a $v \in V$ uniformly at random. Set $\sigma' = \sigma$ with v moved to rank t. Let $u = M(v)$.

According to Lemma 4, if v is not matched under σ' then u is matched under σ to $\tilde{v}$ with $\sigma(\tilde{v}) \leq t$.

Equivalently, $u \in R_t$, where R_t is the subset of vertices U matched to $v' \in V$ s.t. $\sigma(v') \leq t$. In that case, R_t depends only on σ, not on v and u. Therefore,

$$\mathbb{P}[u \in R_t] = \mathbb{E}[\mathbb{P}[u \in R_t]] = \mathbb{E}_{R_t}\left[\frac{|R_t|}{n}\right] = \frac{\sum_{s=1}^{t}}{n}$$

□

Let $S_t = \sum_{s \leq t} x_s$, then $\mathbb{E}[\#\text{ matched vertices}] = \sum_{s \leq n} x_s = S_n$.

2.2.1 Competitive Ratio

According to Lemma 3,

$$1 - x_t \leq \sum_{s \leq t-1} x_s \tag{1}$$

$$\implies 1 - (S_t - S_{t-1}) \leq \frac{S_t}{n} \tag{2}$$

$$\implies S_t \geq \left(1 - \frac{1}{n+1}\right) + \left(1 - \frac{1}{n+1}\right) S_{t-1} \tag{3}$$

Solving the recursion, we get

$$S_0 = 0 \tag{4}$$

$$S_1 \geq 1 - \frac{1}{n+1} \tag{5}$$

$$S_n \geq \sum_{i=1}^{n} \left(1 - \frac{1}{n+1}\right)^i \tag{6}$$

$$= \frac{\left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{n+1}\right)^n}{1/(n+1)} \tag{7}$$

$$\geq (n+1)\left(\frac{n}{n+1} - \frac{1}{e}\right) \tag{8}$$

$$\geq n - \frac{n+1}{e} \tag{9}$$

Thus, the competitive ratio of the ranking algorithm is

$$R(A) = \frac{\mathbb{E}[\#\text{ matched vertices}]}{n} = \frac{S_n}{n} \geq \left(1 - \frac{1}{e}\right) - \frac{1}{en}$$

References

[1] Rajeev Motwani, Prabhakar Raghavan. Randomized Algorithms. *Cambridge University Press*, 0-521-47465-5, 1995.

[2] Karp, Richard M., Umesh V. Vazirani, and Vijay V. Vazirani. An optimal algorithm for on-line bipartite matching. *Proceedings of the twenty-second annual ACM symposium on Theory of computing.* ACM, 1990.