Problem Set 6

Randomized Algorithms

Due Tuesday, December 5

1. You are given a set of vectors $V = v_1, \ldots, v_m \in \mathbb{R}^n$, with all coordinates nonnegative. The goal of this problem is to "sparsify" these vectors: construct $w_1, \ldots, w_k \in \mathbb{R}$ and $u_1, \ldots, u_k \in V$ such that

$$\overline{u} := \sum_{i=1}^{k} w_i u_i$$

and

$$\overline{v} := \sum_{i=1}^{m} v_i$$

satisfy

$$\overline{u}_j = (1 \pm \epsilon)\overline{v}_j$$

for all j.

We would like to solve this using the random sampling techniques developed in class: choose a probability distribution $p_1, \ldots, p_m \in \mathbb{R}$, and draw k independent samples $(w, u) = (\frac{1}{kp_j}, v_j)$ for $j \sim p$.

(a) Show, for some setting of p, that $k = O(n \log n)$ suffices for constant ϵ and all V. What is the dependence on ϵ?

(b) For many inputs V, your result should work with many fewer samples (e.g., $k \ll n$). When does this happen? Give a data-dependent bound for the necessary k as a function of V.

(c) (Optional) Show a different, deterministic, algorithm that uses $k = n$ for all V.

2. Recall our definition of geometric duality in the plane: the dual of a point (a, b) is the line $ax + by = 1$, and vice versa.

   (a) Let p_1, p_2 be two points and l_1, l_2 be their respective dual lines. Show that the line passing between p_1 and p_2 is the dual of the point of intersection of l_1 and l_2.

   (b) Consider a set of points p_i whose convex hull contains the origin. Let l_i denote the dual of p_i, and let h_i be the half-plane bounded by l_i that contains the origin.

   Show that the dual of the convex hull of the p_i is the intersection of the h_i. That is, for any consecutive points $p_{i_1}, p_{i_2}, p_{i_3}$ on the convex hull, show that l_{i_2} contains one side of the polygon representing the boundary of the intersection of the h_i, and that $l_{i_1} \cap l_{i_2}$ and $l_{i_2} \cap l_{i_3}$ are vertices of this polygon.