CS 388R: Randomized Algorithms, Fall 2019 September 19, 2019

Lecture 7: Probability Puzzles

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NOTE: THESE NOTES HAVE NOT BEEN EDITED OR CHECKED FOR CORRECTNESS

1 Balls and Bins Revisit

Question:

1. How many balls can I throws in, typically, into n bins before
   (a) max load is 2 with 1 choice?
   (b) max load is 2 with 2 choices?
2. How many balls can I throws in, typically, into n bins before
   (a) max load is 3 with 1 choice?
   (b) max load is 3 with 2 choices?

(1.a) After k balls are thrown, if max load is 1, then the loads of k bins are 1 and $n-k$ bins are 0. Define failure as the case that the max load is 2. The chance of failure of k-th is $\frac{k}{n}$.

Between $\frac{k}{2}$-th and k-th balls thrown, the failure prob is $O\left(\frac{k}{n}\right)$. The total failure probability after k balls are thrown is approximately $1-\left(1-\frac{k}{n}\right)^k=\Theta\left(\frac{k^2}{n}\right)$.

(1.b) Same analysis as (1.a). Except now the failure probability of k-th is $O(\frac{k^2}{n^2})$. So the total failure probability after k balls are thrown is $1-\left(1-\frac{k^2}{n^2}\right)^k=\Theta\left(\frac{k^3}{n^2}\right)$.

(2.a) After k balls ($k<n$) are thrown,

- expect $\Theta\left(k\right)$ bins with 1 ball.
- expect $\Theta\left(\frac{k}{n}\times k\right)=\Theta\left(\frac{k^2}{n}\right)$ bins with 2 ball.
- with $\Theta\left(\frac{k^3}{n^2}\right)$ probability that there exists a bin with height 3.

Typically, we can throw $\Theta(n^{2/3})$ balls before the max load is 3.

(2.b) After k balls ($k < n$) are thrown,

- expect $\Theta(k)$ bins with 1 ball.
- expect $\Theta\left(\frac{k}{n} \times \frac{k}{n} \times k\right) = \Theta\left(\frac{k^3}{n^2}\right)$ bins with 2 ball.
- with $\Theta\left(\frac{k^7}{n^6}\right)$ probability that there exists a bin with height 3.

Typically, we can throw $\Theta(n^{6/7})$ balls before the max load is 3.

Conclusion: In general, for any height $C = O(1)$. We can throw k balls, typically, before the max load is C where

(a) $k = n^{1-\frac{1}{C}}$ with 1 choice

(b) $k = n^{1-\frac{1}{2^C-1}}$ with 2 choices

2 Random Walk

Question: Start at 0. At each step, move +1 or -1 with equal probability. Stop if reach -10 or 100

(a) Pr[stop at -10]

(b) How long does it stop in expectation?

Solution:

(a)

Method 1: Using **expectation**

Let X_t denote the distribution of its location after moving t steps. $\mathbb{E}[X_t - X_{t-1}] = \frac{1}{2} \times (+1) + \frac{1}{2} \times (-1) = 0$. Therefore, $\mathbb{E}[X_t]$ remains the same for different t. $\mathbb{P}[\text{stops at } -10] \times (-10) + (1 - \mathbb{P}[\text{stops at } 100]) \times 100 = 0$ which indicates $\mathbb{P}[\text{stops at } -10] = \frac{10}{11}$.

Method 2: Let p_i denote the probability of stopping at -10 when starting at i. We have

$$p_i = \begin{cases} 1 & \text{if } i = -10; \\ 0 & \text{if } i = 100; \\ \frac{1}{2}\left(p_{i-1} + p_{i+1}\right) & \text{otherwise.} \end{cases}$$

By solving this equation, We have p_i is linear with i and $p_0 = \frac{10}{11}$.

(b)

Let n_i denote the expected number of steps to stop when starting at i. We have

$$n_i = \begin{cases} 0 & \text{if } i = -10 \text{ or } 100; \\ \frac{1}{2}\left(n_{i-1} + 1\right) + \frac{1}{2}\left(n_{i+1} + 1\right) & \text{otherwise.} \end{cases}$$

By solving this equation, we have $n_i = -i^2 + 90i + 1000$ and $n_0 = 1000$.