Problem Set 4

Sublinear Algorithms

Due Tuesday, November 8

1. Give an algorithm to construct the k-tree-sparse approximation of a vector. The input is an integer k and a complete n-vertex binary tree T with a nonnegative value x_v associated with each vertex v. The output is the set S of size k that includes the root, is a connected subset of the tree, and maximizes $\sum_{v \in S} x_v$.

   (a) Let s_v be the size of the subtree rooted at v, and let $l(v)$, $r(v)$ denote the left/right children of v (if any). Show a DP algorithm for tree sparsification that runs in order

   $$\sum_{v \in T} \min(1 + s_{l(v)}, k) \cdot \min(1 + s_{r(v)}, k)$$

   time. Note that this is trivially $O(nk^2)$.

   (b) Show that this is actually $O(nk)$ on any (not necessarily complete) binary tree. Hint: split into three cases, corresponding to the nodes v where zero, one, or two of the children have at least k descendents.

2. In this problem, we show a lower bound for randomized ℓ_2/ℓ_2 *adaptive* compressed sensing with *Fourier* measurements. Let $F \in \mathbb{C}^{n \times n}$ be a fixed orthogonal matrix with $|F_{i,j}| = 1$ for all i, j.

   An *adaptive recovery algorithm* with measurements from F will, for an unknown approximately sparse vector x, repeatedly choose an index i_t and see $z_t = (Fx)_{i_t}$. Each choice of index may depend on previous observations. After m observations, the algorithm should return a vector $\widehat{x}$ with

   $$\|\widehat{x} - x\|_2 \leq C \min_{k\text{-sparse } x_k} \|x - x_k\|_2$$

with 3/4 probability, for some constant C. We will show that such an algorithm must have $m = \Omega(k \log(n/k)/ \log\log n)$.

(a) Show that there exists a set $S \subset \mathbb{R}^n$ of k-sparse vectors such that (I) $\|x\|_2 = \sqrt{k}$ for all $x \in S$, (II) $\|x - y\|_2 \geq \sqrt{k}$ for all $x \neq y$, $x, y \in S$, (III) $\|Fx\|_\infty \lesssim \sqrt{k \log n}$ for all $x \in S$, and (IV) $\log |S| \gtrsim k \log(n/k)$. (Hint: Recall the large set $S \subset \{0,1\}^n$ of k-sparse vectors from class, and flip the signs randomly.)

(b) Consider choosing a random $x \in S$ and $w \sim N(0, \sigma^2 I_n)$ for $\sigma < \frac{1}{100C}\sqrt{k/n}$, and setting $x' = x + w$. Show that successful sparse recovery $\widehat{x}'$ of x' is sufficient to uniquely identify x.

(c) Show that this implies $I(\widehat{x}'; x) \gtrsim k \log(n/k)$.

(d) Separately, use the Shannon-Hartley theorem to show that, for any fixed i,
$$I((Fx')_i; x) \lesssim \log\log n.$$

(e) Show that this implies
$$I((z_1, \ldots, z_m); x) \lesssim m \log\log n.$$
(Hint: extend the lemma we proved for linear measurements in the context of moment bounds to the adaptive setting.)

(f) Conclude that $m \gtrsim k \log(n/k)/ \log\log n$.

3. Final project. Spend at least one page describing your planned final project: what you plan to do, who you're working with, and a proposed schedule.