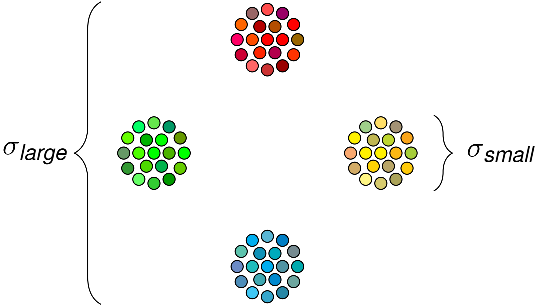
RIP of Subsampled Fourier Matrix

Based off Rudelson-Vershynin

Eric Price

2020-10-27



Outline

1. Introduction
   - Compressive sensing
   - Johnson Lindenstrauss Transforms

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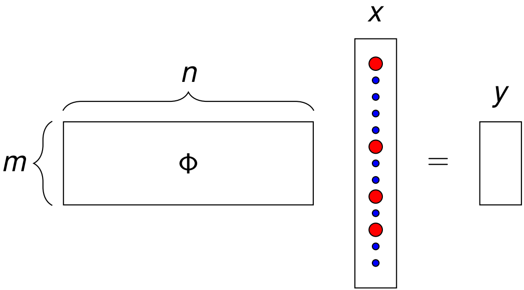
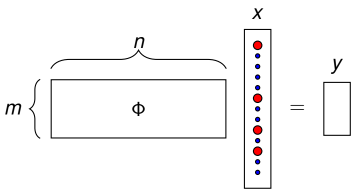
Compressive Sensing

Given: A few linear measurements of an (approximately) k-sparse vector $x \in \mathbb{R}^n$.
Goal: Recover x (approximately).



Compressive Sensing Algorithms: Two Classes



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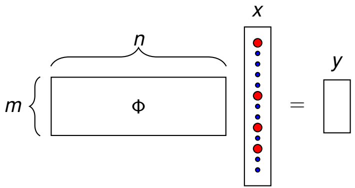

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Recovery algorithm tied to matrix structure (e.g. Count-Sketch)

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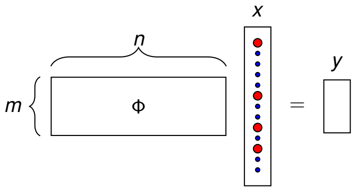

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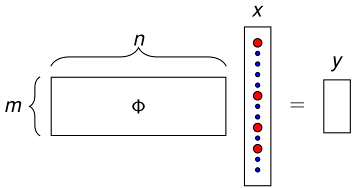

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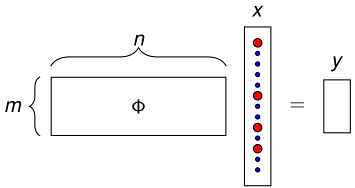

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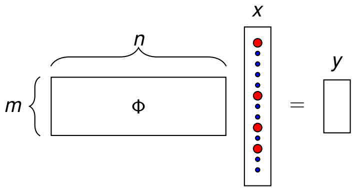

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- For all of these:
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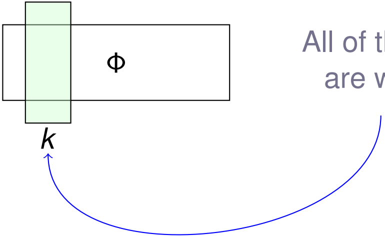
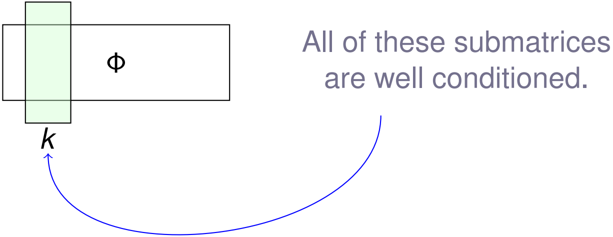
  - the *Restricted Isometry Property* is a sufficient condition.

Restricted Isometry Property (RIP)



All of these submatrices
are well conditioned.

Restricted Isometry Property (RIP)



$$(1-\epsilon)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1+\epsilon)\|x\|_2^2$$

for all k-sparse $x \in \mathbb{R}^n$.

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  - Random Gaussian matrix: $\Theta(nk \log n)$ time.
- Goal: an RIP matrix with $O(n \log n)$ multiplication and small m.

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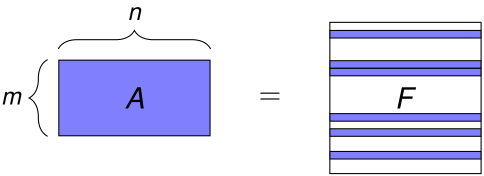
An open question



Let A contain random rows from a Fourier matrix.

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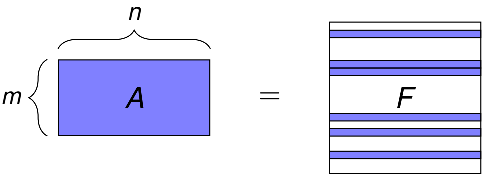

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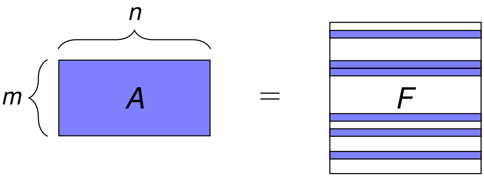

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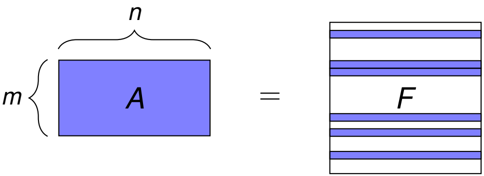

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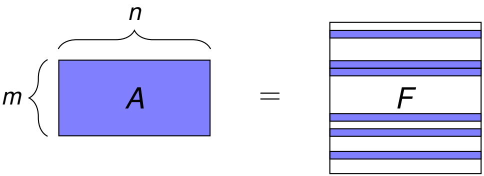

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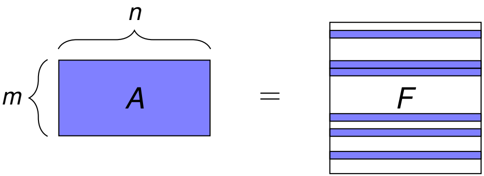

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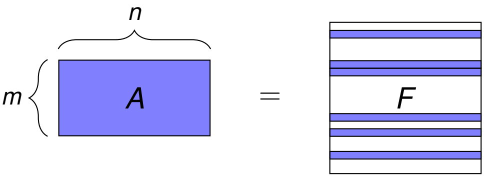

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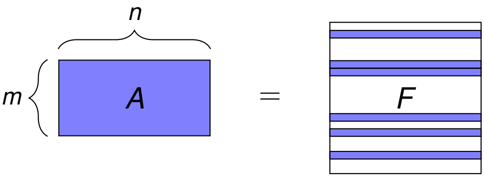

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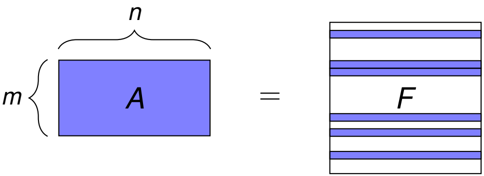

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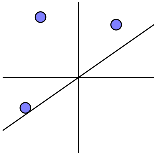
(Related: how about partial circulant matrices?)

- $m = O(k \log^4 n)$ [RRT12,KMR12].

Outline

Another motivation: Johnson Lindenstrauss (JL) Transforms



High dimensional data
$S \subset \mathbb{R}^n$

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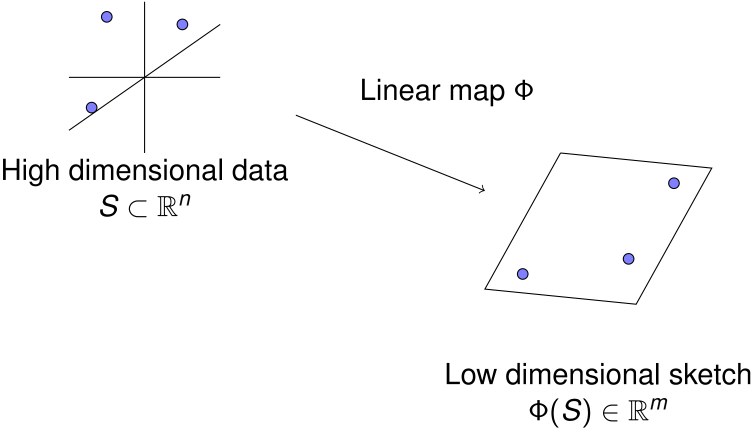

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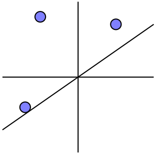

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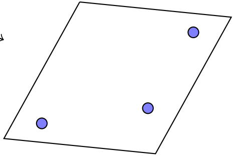

Φ preserves the geometry of S

Low dimensional sketch
$\Phi(S) \in \mathbb{R}^m$

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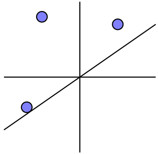

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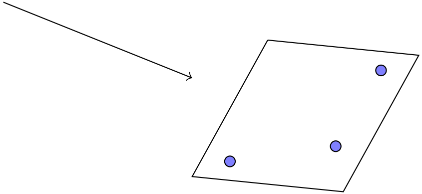

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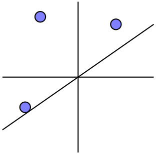

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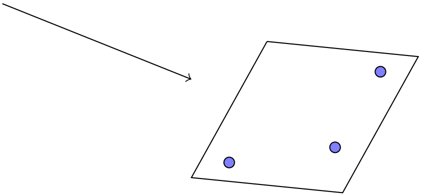

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$$\langle \Phi x, \Phi y\rangle = \langle x, y\rangle \pm \epsilon \|x\|_2 \|y\|_2$$

Johnson-Lindenstrauss Lemma

Theorem (variant of Johnson-Lindenstrauss '84)

Let $x \in \mathbb{R}^n$. A random Gaussian matrix Φ will have

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Set $\delta = 1/2^k$: embed 2^k points into $O(k)$ dimensions.

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- Fast multiplication.
  - Approximate numerical algebra problems (e.g., linear regression, low-rank approximation)
  - k-means clustering

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  - $n \log n$ multiplication time.
- And by [BDDW '08], JL $\Rightarrow$ RIP; so *equivalent*.[1]

[1] Round trip loses $\log n$ factor in dimension

Concentration of Measure

Let Σ_k is unit-norm k-sparse vectors.
We want to show for our distribution Φ on matrices that

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Probabilists have lots of tools to analyze this.

Outline

Tools

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Screwdriver

Tools

Screwdriver

Drill

Tools

Screwdriver

Bit sets

Drill

Tools

Screwdriver

Drill

Bit

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Drill

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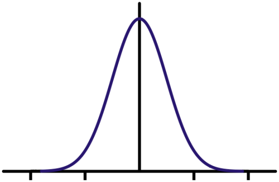


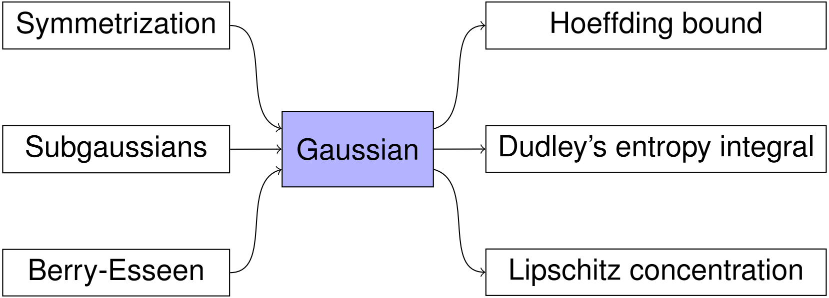
Gaussians

Common interface
for probability

A Probabilist's Toolbox

Convert to Gaussians

Gaussian concentration



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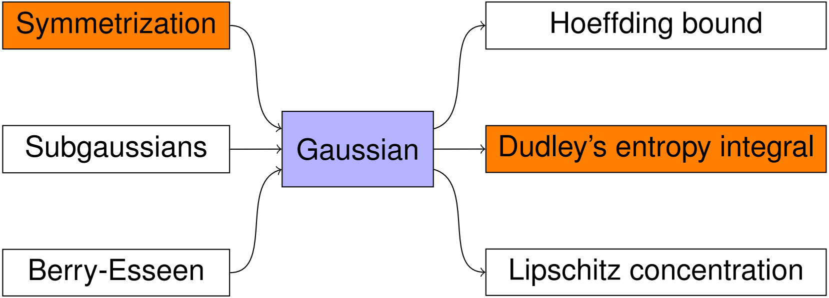

Will prove: symmetrization and Dudley's entropy integral.

Outline

Symmetrization

Lemma (Symmetrization)

Suppose $X_1, \ldots, X_t$ are i.i.d. with mean μ. For any norm $\|\cdot\|$,

$$\mathbb{E}\left[\left\|\frac{1}{t}\sum_i X_i - \mu\right\|\right] \leq 2\,\mathbb{E}\left[\left\|\frac{1}{t}\sum_i s_i X_i\right\|\right]$$

where $s_i \in \{\pm 1\}$ independently.

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Suppose $X_1, \dots, X_t$ are i.i.d. with mean μ. For any norm $\|\cdot\|$,

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and apply the triangle inequality. $\square$

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□

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Gaussian Processes

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Example (Maximum singular value of random Gaussian matrix)

Let A be a random $m \times n$ Gaussian matrix. For any $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, define

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Then $G_{u,v} \sim N(0, \|uv^T\|_F^2)$.

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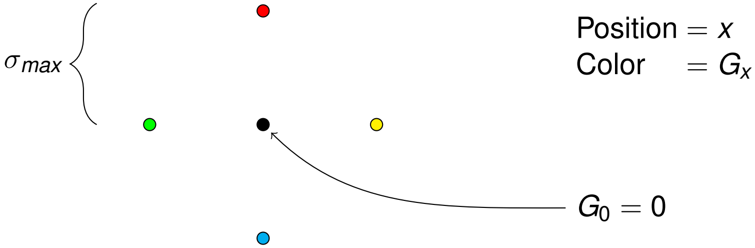

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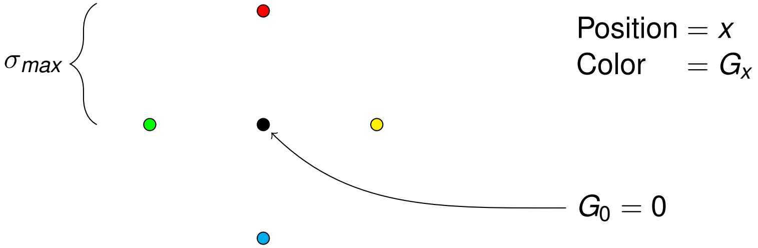

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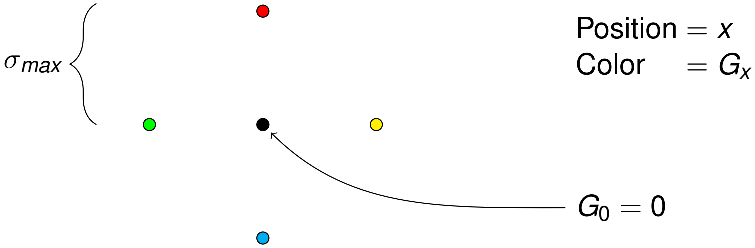

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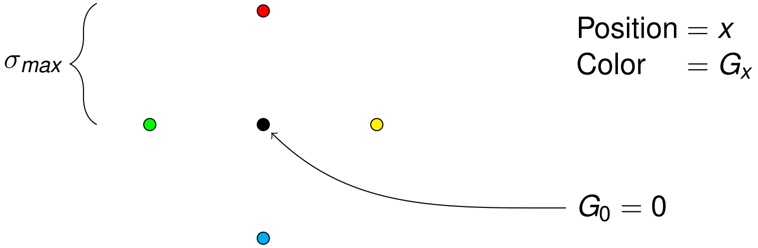

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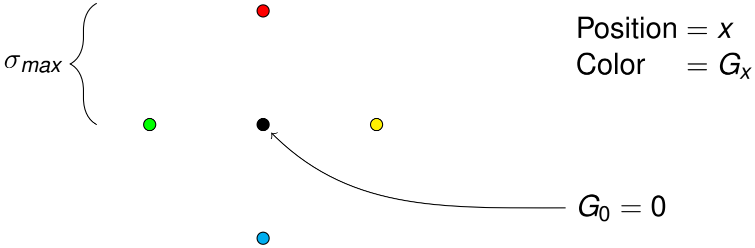

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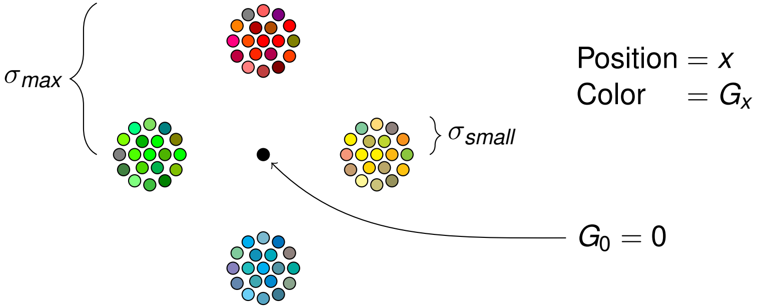

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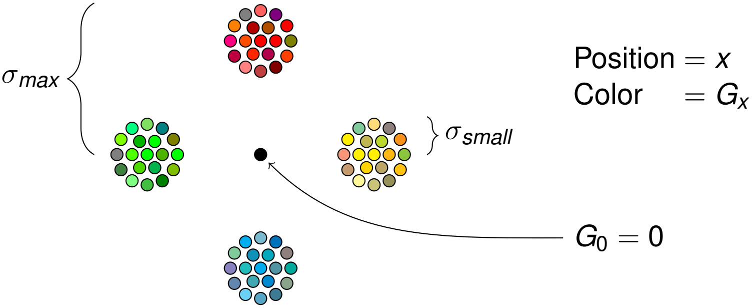

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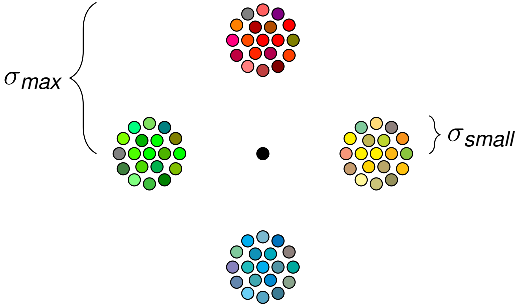

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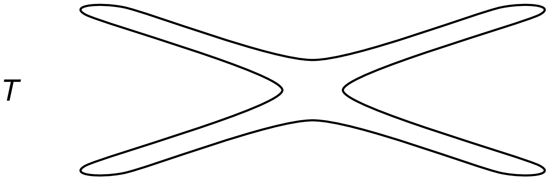

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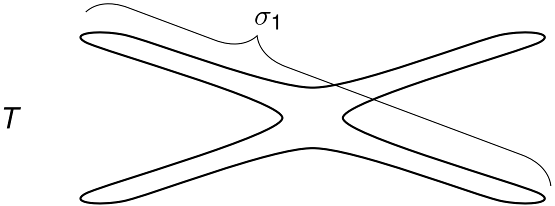

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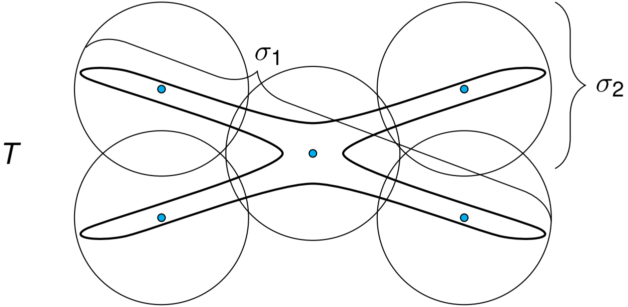

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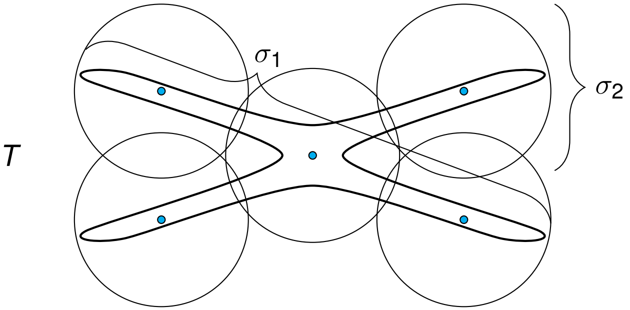

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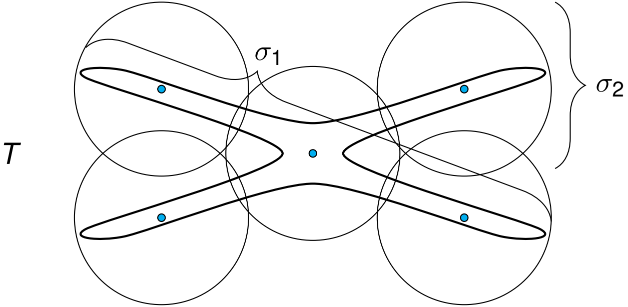

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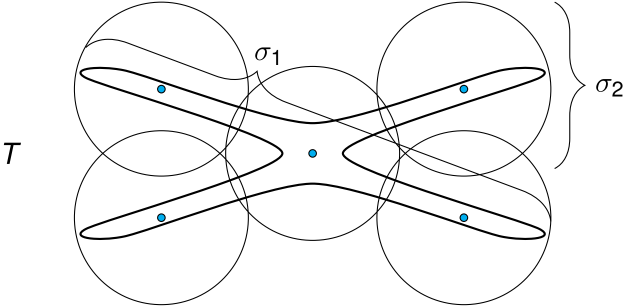

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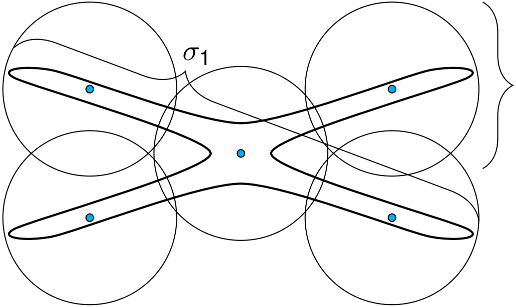

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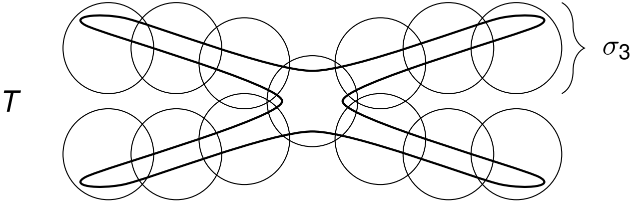

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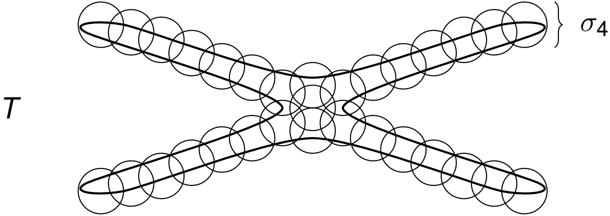

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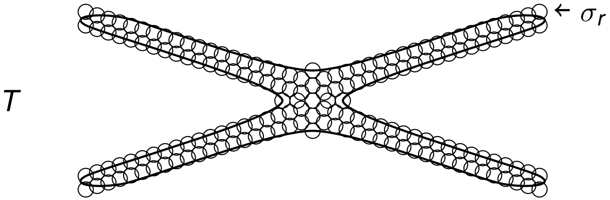

Gaussian Processes: chaining

- Bound $\mathbb{E}\sup_{x\in T} G_x$, where $G_x - G_y$ has variance $\|x-y\|^2$.
- Two levels: $\sigma_1\sqrt{\log N(\sigma_2)} + \sigma_2\sqrt{\log n}$.
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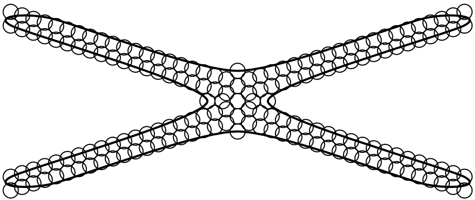
T



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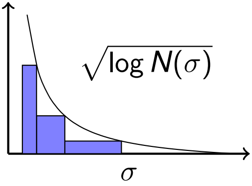

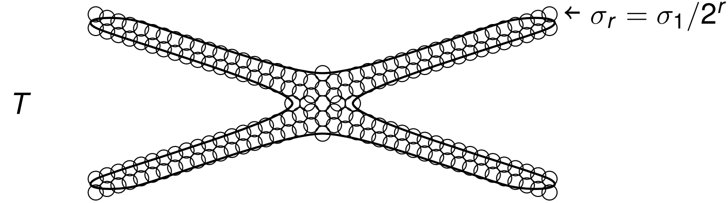

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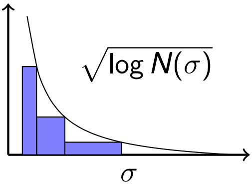

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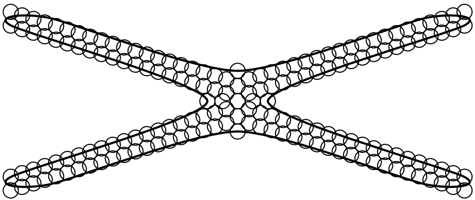

$\leftarrow \sigma_r = \sigma_1/2^r$

Gaussian Processes

Dudley's Entropy Integral, Talagrand's generic chaining

Theorem (Dudley's Entropy Integral)

Define the norm $\|\cdot\|$ of a Gaussian process G by

$$\|x - y\| = \text{standard deviation of } (G_x - G_y).$$

Then

$$\mathbb{E} \sup_{x \in T} G_x \lesssim \int_0^\infty \sqrt{\log N(T, \|\cdot\|, u)} du$$

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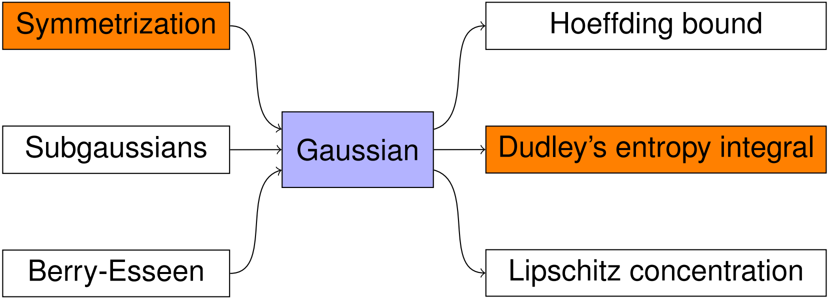
- Bound a random variable using geometry.

A Probabilist's Toolbox (recap)

Convert to Gaussians

Gaussian concentration



Outline

Goal

Let $\Omega \subset [n]$ have each $i \in [n]$ independently with probability m/n. Let

$$A = \frac{1}{\sqrt{m}} F_\Omega$$

For Σ_k denoting unit-norm k-sparse vectors, we want

$$\mathop{\mathbb{E}}_{\Omega} \sup_{x \in \Sigma_k} \left| \|Ax\|_2^2 - \|x\|_2^2 \right| < \epsilon,$$

(Expectation of $\color{red}{*}$) = $\color{blue}{*}$

Proof outline: Rudelson-Vershynin

Rudelson-Vershynin: subsampled Fourier, $O(k \log^4 n)$ rows.

$$\mathbb{E} \sup_{x} x^T(A^T A - \mathrm{I})x$$

Expected
sup deviation

Proof outline: Rudelson-Vershynin

Rudelson-Vershynin: subsampled Fourier, $O(k \log^4 n)$ rows.

$\mathbb{E} \sup$
$x^T(A^T A - \mathrm{I})x$

Expected sup deviation

Symmetrization

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$\mathbb{E} \sup x^T(A^T A - \mathrm{I})x$ — Expected sup deviation

↓

Symmetrization

↓

$\gamma_2(\Sigma_k, \|\cdot\|)$ — Expected sup norm of Gaussian

γ_2 : supremum of Gaussian process

Σ_k : k-sparse unit vectors

$\|\cdot\|$: a norm that depends on A

(specified in a few slides)

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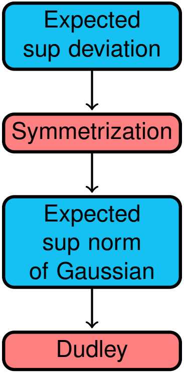

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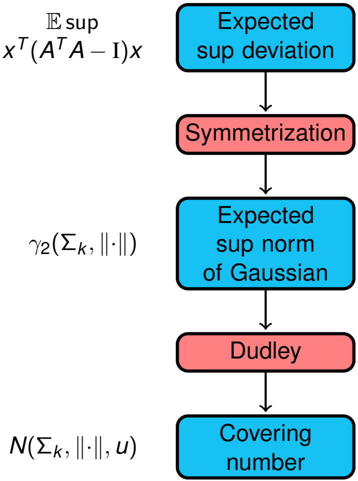

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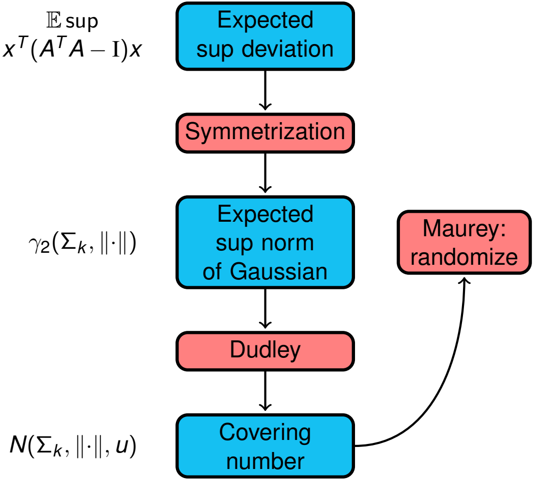

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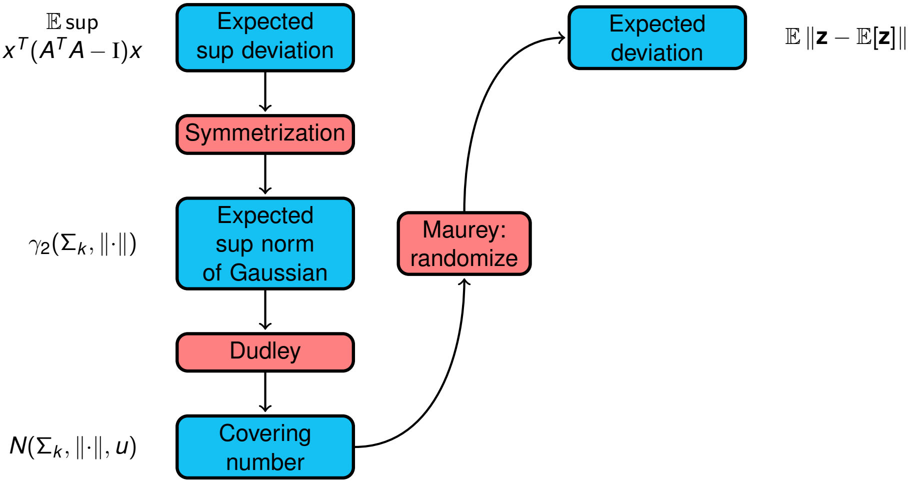

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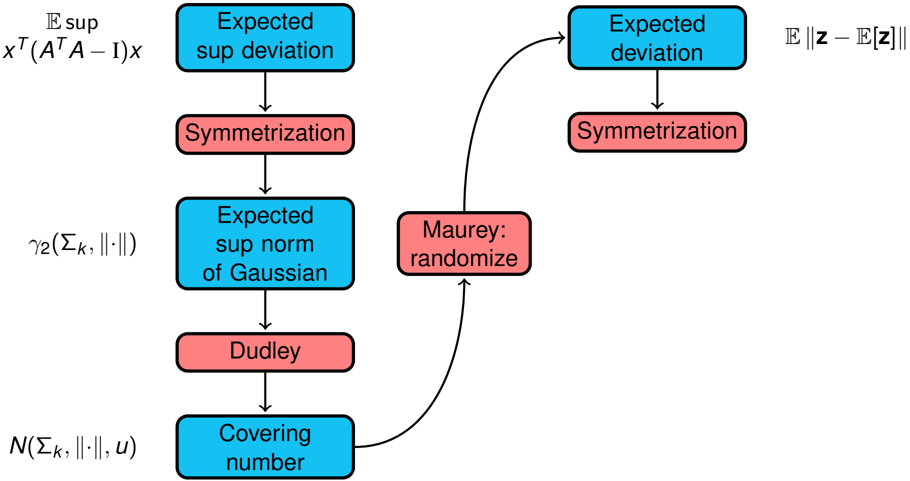

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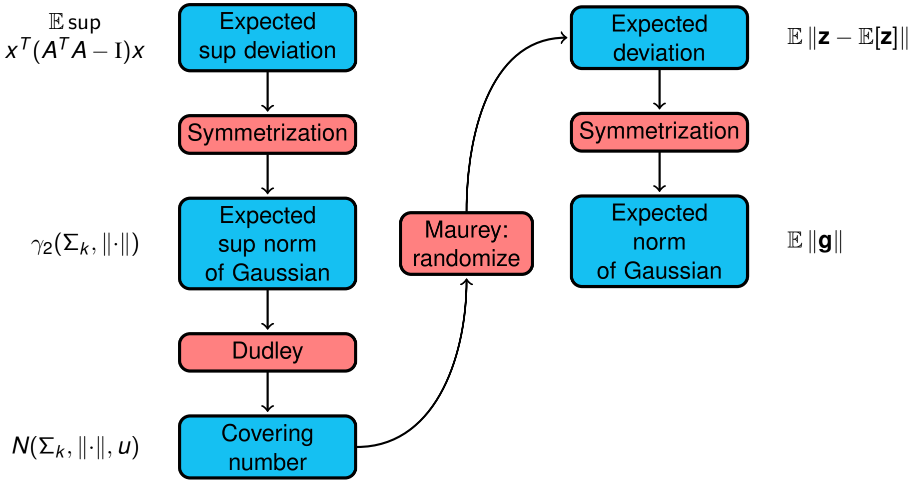

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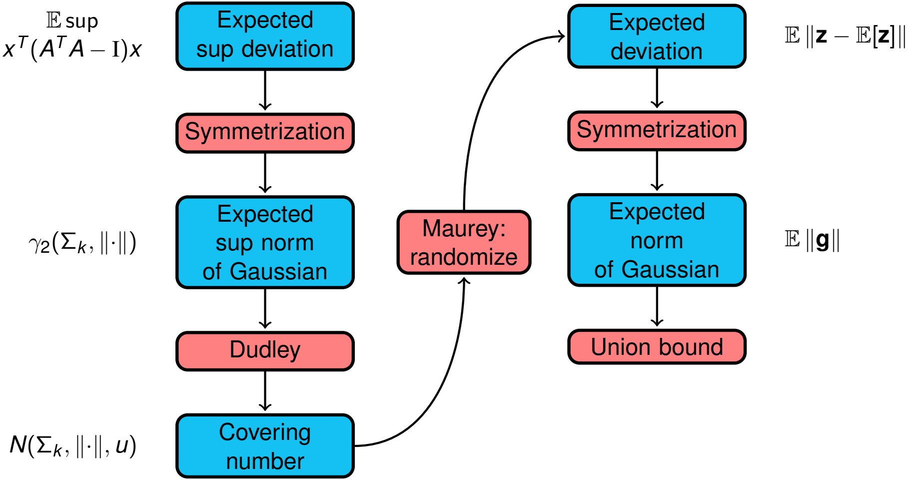

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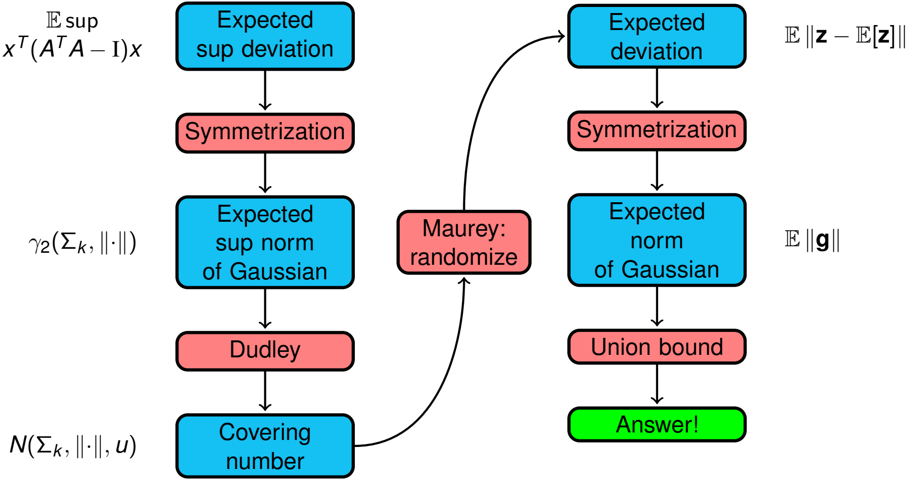

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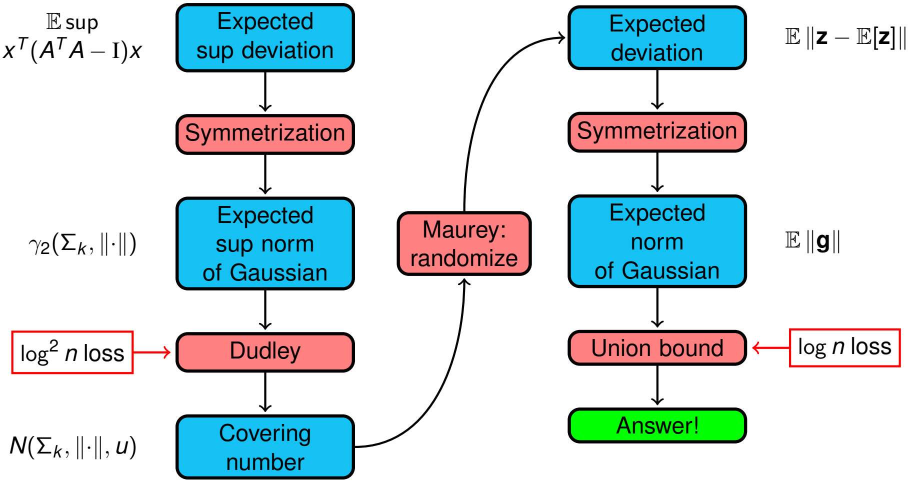

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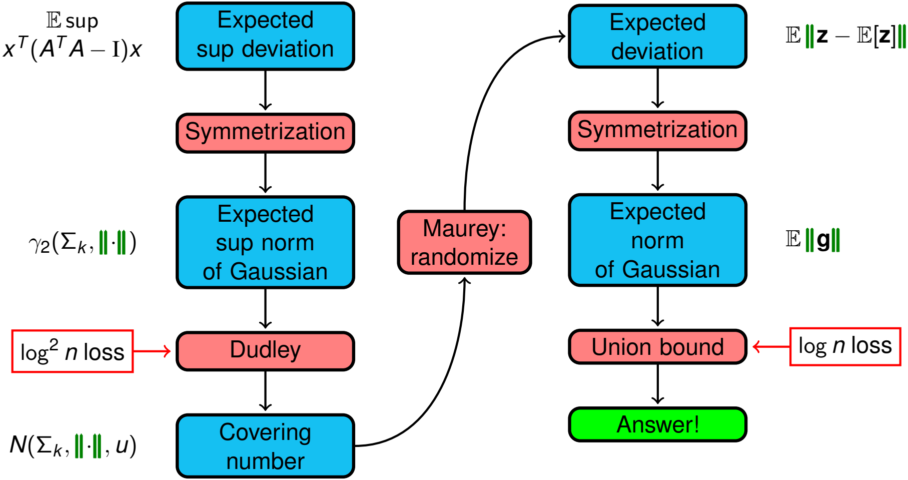


Setup

Let $\delta_i = 1_{i\in\Omega}$. Then $\Pr[\delta_i] = m/n$, independently for all i.

$$Ax = \frac{1}{\sqrt{m}} \sum_{i=1}^{n} \delta_i F_i x.$$

where $F_{ij} = e^{2\pi\sqrt{-1}ij/n}$. We would like to analyze the RIP constant

$$R_\Omega := \sup_{x\in\Sigma_k} |x^T A^T A x - 1|.$$

Now, for any fixed x,

$$\mathop{\mathbb{E}}_{\Omega}[x^T A^T A x] = \frac{1}{n} x^T F^T F x = \|x\|_2^2$$

and hence

$$\mathop{\mathbb{E}}_{\Omega}[R_\Omega] = \mathop{\mathbb{E}}_{\Omega} \sup_{x\in\Sigma_k} \left| \|Ax\|_2^2 - \|x\|_2^2 \right| = \mathop{\mathbb{E}}_{\Omega} \sup_{x\in\Sigma_k} \left| \|Ax\|_2^2 - \mathop{\mathbb{E}}_{\Omega}[\|Ax\|_2^2] \right|$$

Proof part 1: symmetrization

$$\begin{aligned}
\mathbb{E}[R_\Omega] &= \mathop{\mathbb{E}}_{\Omega} \sup_{x \in \Sigma_k} \left| \|Ax\|_2^2 - \|x\|_2^2 \right| \\
&= \frac{1}{m} \mathop{\mathbb{E}}_{\delta} \sup_{x \in \Sigma_k} \left| \sum_{i=1}^n \delta_i \langle F_i, x\rangle^2 - \mathbb{E}[\sum_{i=1}^n \delta_i \langle F_i, x\rangle^2] \right| \\
&\leq \frac{1}{m} 3 \mathop{\mathbb{E}}_{\delta, g} \sup_{x \in \Sigma_k} \left| \sum_{i=1}^n g_i \delta_i \langle F_i, x\rangle^2 \right| \\
&\leq \frac{1}{m} 3 \mathop{\mathbb{E}}_{\Omega} \mathop{\mathbb{E}}_{g} \sup_{x \in \Sigma_k} \left| \sum_{i \in \Omega} g_i \langle F_i, x\rangle^2 \right| .
\end{aligned}$$

The Gaussian Process

So by symmetrization,

$$m\,\mathbb{E}[R_\Omega] \lesssim \mathbb{E}_{\Omega}\,\mathbb{E}_{g} \sup_{x\in\Sigma_k} \left|\sum_{i\in\Omega} g_i\langle F_i, x\rangle^2\right|.$$

Now fix Ω, and define the Gaussian process

$$G_x = \sum_{i\in\Omega} g_i\langle F_i, x\rangle^2.$$

which induces the norm

$$\|x-y\|_G^2 = \mathbb{E}[(G_x - G_y)^2] = \sum_{i\in\Omega}(\langle F_i, x\rangle^2 - \langle F_i, y\rangle^2)^2$$

so that

$$m\,\mathbb{E}[R] \lesssim \mathbb{E}_{\Omega}\,\mathbb{E} \sup_{x\in\Sigma_k} G_x =: \mathbb{E}_{\Omega}\,\gamma_2(\Sigma_k, \|\cdot\|_G)$$

$$\leq \mathbb{E}_{\Omega} \int_0^\infty \sqrt{\log N(\Sigma_k, \|\cdot\|_G, u)du}$$

by Dudley's entropy integral.

Simplifying the norm

$$\begin{aligned}
\|x-y\|_G^2 &= \sum_{i\in\Omega}(\langle F_i, x\rangle^2 - \langle F_i, y\rangle^2)^2 \\
&= \sum_{i\in\Omega}(\langle F_i, x+y\rangle \cdot \langle F_i, x-y\rangle)^2 \\
&\leq (\sum_{i\in\Omega}\langle F_i, x+y\rangle^2)\max_{i\in\Omega}\langle F_i, x-y\rangle^2 \\
&\leq (4\sup_{x'\in\Sigma_k}\sum_{i\in\Omega}\langle F_i, x'\rangle^2)\max_{i\in[n]}\langle F_i, x-y\rangle^2 \\
&\leq 4m(1+R_\Omega)\|F(x-y)\|_\infty^2.
\end{aligned}$$

If we define $\|x\|_F := \|Fx\|_\infty$, this means

$$\begin{aligned}
m\,\mathbb{E}[R_\Omega] &\lesssim \mathop{\mathbb{E}}_{\Omega}\int_0^\infty \sqrt{\log N(\Sigma_k, \|\cdot\|_F, u/\sqrt{1+R_\Omega})du} \\
&\leq \mathop{\mathbb{E}}_{\Omega}\sqrt{m(1+R_\Omega)}\int_0^\infty \sqrt{\log N(\Sigma_k, \|\cdot\|_F, u)du}
\end{aligned}$$

New goal

Have:

$$m\,\mathbb{E}[R_\Omega] \lesssim \left(\mathop{\mathbb{E}}_{\Omega}\sqrt{m(1+R_\Omega)}\right)\int_0^\infty \sqrt{\log N(\Sigma_k, \|\cdot\|_F, u)du}$$

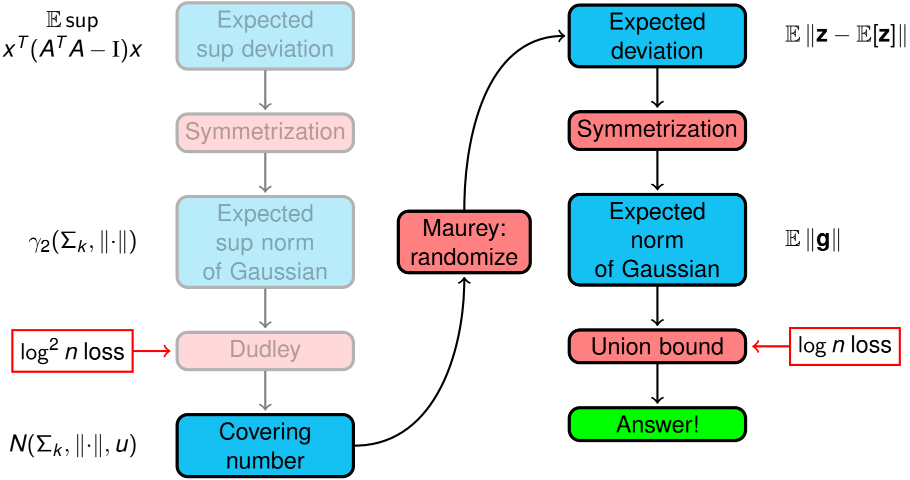
Will show:

$$\int_0^\infty \sqrt{\log N(\Sigma_k, \|\cdot\|_F, u)du} \lesssim \epsilon\sqrt{m}$$

for $\epsilon < 1$. This implies that $\mathbb{E}[R_\Omega] \lesssim \epsilon\,\mathbb{E}[\sqrt{1+R_\Omega}]$, and hence

$$\mathbb{E}[R_\Omega] \lesssim \epsilon.$$

Progress

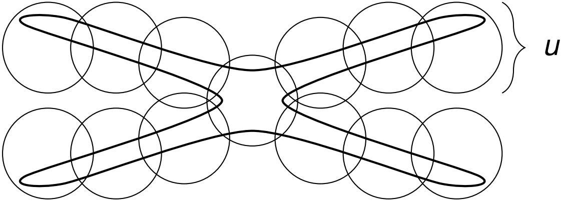


$\mathbb{E}\sup_x x^T(A^TA - I)x$
Expected sup deviation
Symmetrization
$\gamma_2(\Sigma_k, \|\cdot\|)$
Expected sup norm of Gaussian
$\log^2 n$ loss
Dudley
$N(\Sigma_k, \|\cdot\|, u)$
Covering number
Maurey: randomize
Expected deviation
$\mathbb{E}\|\mathbf{z} - \mathbb{E}[\mathbf{z}]\|$
Symmetrization
Expected norm of Gaussian
$\mathbb{E}\|\mathbf{g}\|$
Union bound
$\log n$ loss
Answer!

Outline

Covering Number Bound

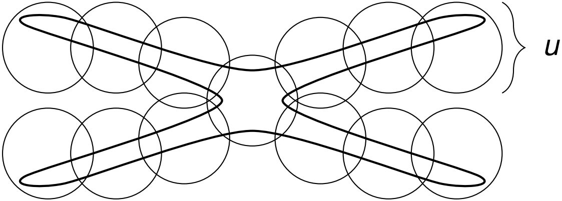
$$N(\Sigma_k, \|\cdot\|_F, u)$$



$$\Sigma_k = \{k\text{-sparse } x \mid \|x\|_2 \le 1\}$$

Covering Number Bound

$$N(\Sigma_k, \|\cdot\|_F, u) \leq N(B_1, \|\cdot\|_F, u/\sqrt{k})$$



$$\begin{aligned} \Sigma_k &= \{k\text{-sparse } x \mid \|x\|_2 \leq 1\} \\ \subset \sqrt{k}B_1 &= \{x \mid \|x\|_1 \leq \sqrt{k}\} \end{aligned}$$

Covering number bound

$$N(B_1, \|\cdot\|_F, u)$$

Covering number bound

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- Simpler to imagine: what about ℓ_2?

Covering number bound

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- How many ℓ_2 balls of radius u required to cover B_1?

$$N(B_1, \|\cdot\|_2, u)$$

Covering number bound

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- Simpler to imagine: what about ℓ_2?
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$$N(B_1, \|\cdot\|_2, u) \lesssim \begin{cases} (1/u)^{O(n)} & \text{by an easy volume argument} \\ & \end{cases}$$

Covering number bound

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- *Maurey's empirical method*: generalizes to arbitrary norms

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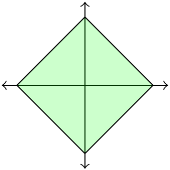
Covering Number Bound

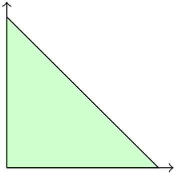
Maurey's empirical method



- How many balls of radius u required to cover B_1 ?

Covering Number Bound

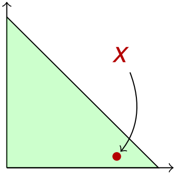
Maurey's empirical method



- How many balls of radius u required to cover B_1^+?

Covering Number Bound

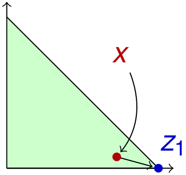
Maurey's empirical method



- How many balls of radius u required to cover B_1^+?
- Consider any $x \in B_1^+$.

Covering Number Bound

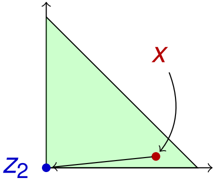
Maurey's empirical method



- How many balls of radius u required to cover B_1^+?
- Consider any $x \in B_1^+$.
- Let $z_1, \ldots, z_t$ be i.i.d. randomized roundings of x to simplex.

Covering Number Bound

Maurey's empirical method



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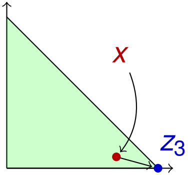

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Covering Number Bound

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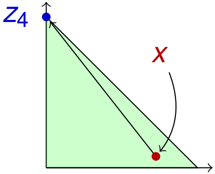

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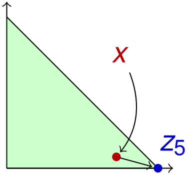

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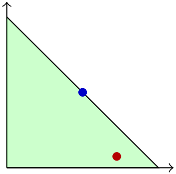
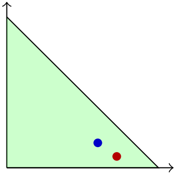


- How many balls of radius u required to cover B_1^+?
- Consider any $x \in B_1^+$.
- Let $z_1, \ldots, z_t$ be i.i.d. randomized roundings of x to simplex.
- The sample mean $\mathbf{z} = \frac{1}{t}\sum z_i$ converges to x as $t \to \infty$.

Covering Number Bound

Maurey's empirical method



- How many balls of radius u required to cover B_1^+?
- Consider any $x \in B_1^+$.
- Let $z_1, \ldots, z_t$ be i.i.d. randomized roundings of x to simplex.
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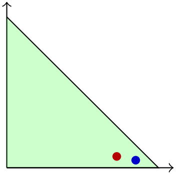

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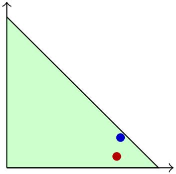

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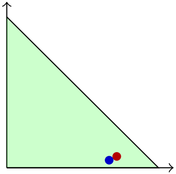

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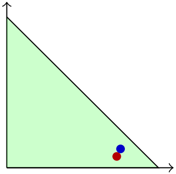

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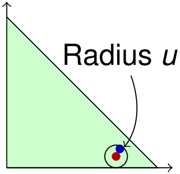

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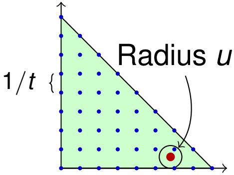

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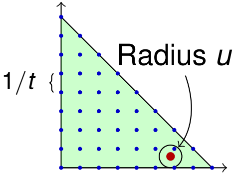

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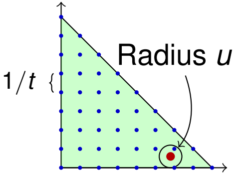

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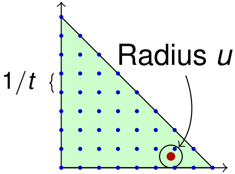

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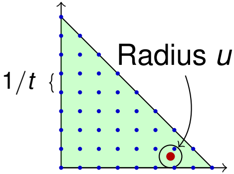

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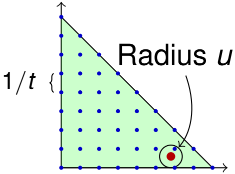

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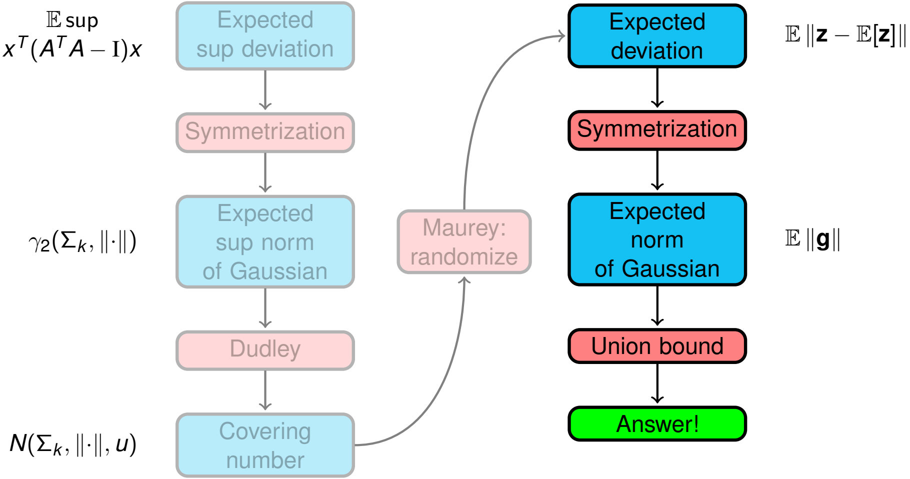
- Thus $t = \log n/u^2$ suffices, for $N(B_1, \|\cdot\|_F, u) \le n^{O(\log n)/u^2}$.

Progress



$\mathbb{E} \sup x^T(A^TA - \mathrm{I})x$
Expected sup deviation
Symmetrization
$\gamma_2(\Sigma_k, \|\cdot\|)$
Expected sup norm of Gaussian
Dudley
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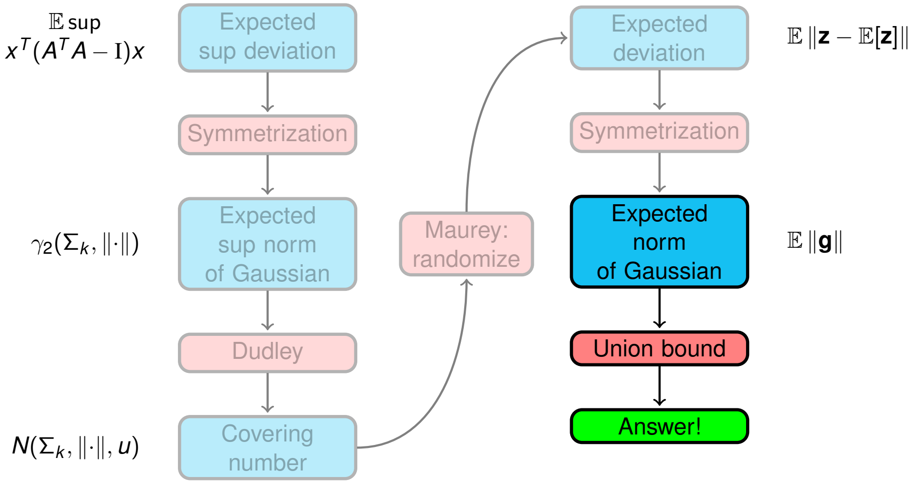
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$$\log N(B_1, \|\cdot\|_F, u) \lesssim \frac{1}{u^2} \log^2 n$$
$$\log N(\Sigma_k, \|\cdot\|_F, u) \lesssim \frac{k}{u^2} \log^2 n$$

And hence

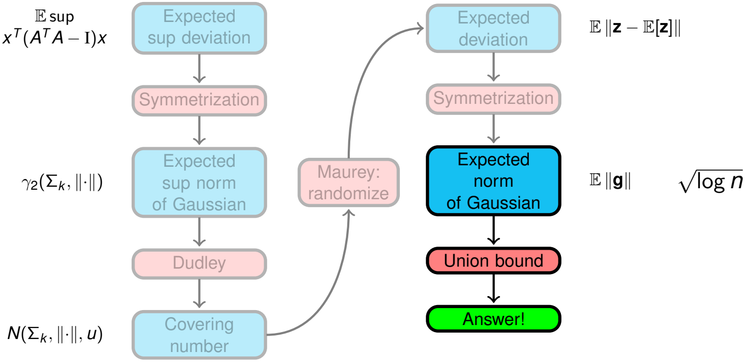
$$\int_{1/n^{100}}^{n} \sqrt{\log N(\Sigma_k, \|\cdot\|_F, u)} du \lesssim \sqrt{k \log^4 n} \leq \epsilon \sqrt{m}$$

if $m \geq \frac{1}{\epsilon^2} k \log^4 n$, which is what we needed.
[Note: Small u are negligible by the volume argument:

$$\int_0^{1/n^{100}} \sqrt{\log N(\Sigma_k, \|\cdot\|_F, u)} du \lesssim \sqrt{k \log^2 n} \int_0^{1/n^{100}} n \log(1/u) du \ll 1/n^{97}$$

]

Unrolling everything



Union bound of $[n]$ uses $\log n$ factor

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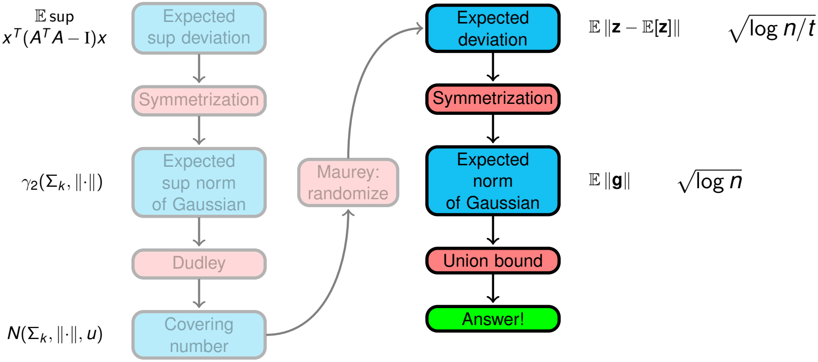

Sample mean $\mathbf{z}$ expects to lie within u of $\mathbf{x}$ for $t \geq \log n/u^2$

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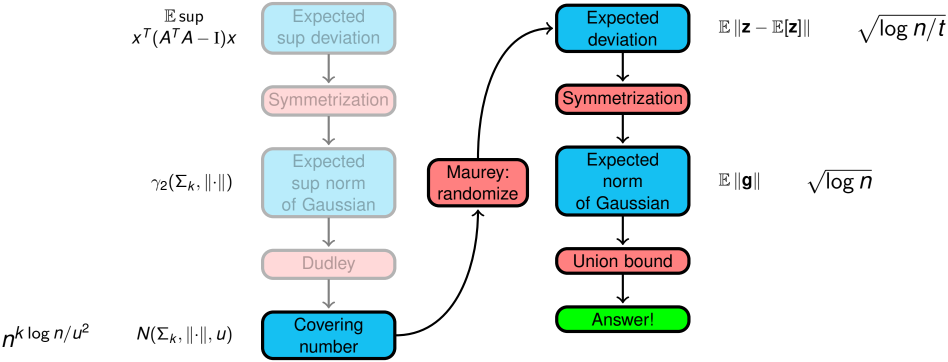

Covering number of B_1 is $(n+1)^{\log n/u^2}$

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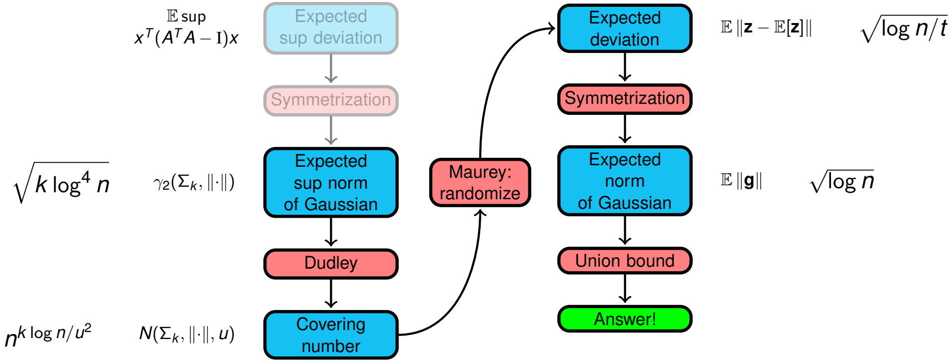


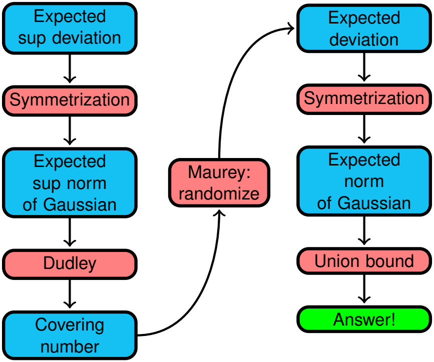
Entropy integral is $\sqrt{\frac{k\log^4 n}{m}}$

Unrolling everything

$\sqrt{\frac{k \log^4 n}{m}}$ $\quad \mathbb{E} \sup x^T(A^TA - \mathrm{I})x$

$\sqrt{k \log^4 n}$ $\quad \gamma_2(\Sigma_k, \|\cdot\|)$

$n^{k \log n / u^2}$ $\quad N(\Sigma_k, \|\cdot\|, u)$



$\mathbb{E}\|\mathbf{z} - \mathbb{E}[\mathbf{z}]\|$ $\quad \sqrt{\log n/t}$

$\mathbb{E}\|\mathbf{g}\|$ $\quad \sqrt{\log n}$

$$\text{RIP constant } \epsilon \lesssim \sqrt{\frac{k \log^4 n}{m}}$$

Summary

- Symmetrization and covering numbers are very general tools!

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- Dudley: choose A_i so $\sup d(x, A_i) \le \sigma_1 / 2^i$.

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- In ℓ_2,

$$\frac{1}{t}\mathbb{E}[\|g\|_2] \le \frac{1}{t}\mathbb{E}[\|g\|_2^2]^{1/2} = \frac{\sqrt{\text{number nonzero } z_i}}{t} \le \frac{1}{\sqrt{t}}.$$

  giving an $n^{O(1/u^2)}$ bound.

Bounding the norm in our case (part 1)

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$$\mathcal{G}(x) = \mathop{\mathbb{E}}_{z,g} \|g\|_A$$

- First: split x into "large" and "small" coordinates.

$$\mathcal{G}(x) \leq \mathcal{G}(x_{large}) + \mathcal{G}(x_{small})$$

- x_{large}: Locations where $x_i > (\log n)/k$
  - Bound:

$$\|x_{large}\|_1$$

  - Given $\|x\|_2^2 \leq 1/k$, maximal $\|x_{large}\|_1$ if spread out.

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- So can focus on $\|x\|_\infty < (\log n)/k$.

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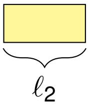

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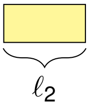

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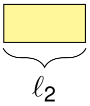

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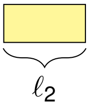

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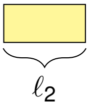

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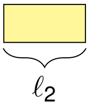

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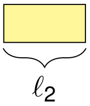

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- So with high probability, $\|A_i g\|_2 \lesssim \sqrt{B/t} + C\sqrt{\log n} \lesssim \sqrt{B/t}$.
- So $\mathbb{E}\|g\|_A = \max \|A_i g\|_2 \lesssim \sqrt{B/t}$.