Problem Set 8

Sublinear Algorithms

Due Thursday, November 12

1. In class we have shown various algorithms for sparse recovery that tolerate noise and use $O(k \log(n/k))$ measurements, and shown that any $\ell_1/\ell_1$ sparse recovery algorithm must use this many measurements. But what if we don’t care about tolerating noise, and only want to recover $x$ from $Ax$ when $x$ is exactly $k$-sparse?

Consider the matrix

$$A = \begin{pmatrix}
\alpha^1_1 & 1 & \cdots & \alpha^1_n \\
\alpha^2_1 & \alpha^2_2 & \cdots & \alpha^2_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^{2k-1}_1 & \alpha^{2k-1}_2 & \cdots & \alpha^{2k-1}_n
\end{pmatrix}$$

for distinct $\alpha^i_i$.

(a) Prove that any $2k \times 2k$ submatrix of $A$ is invertible. (Hint: look up the Vandermonde determinant.)

(b) Give an $n^{O(k)}$ time algorithm to recover $x$ from $Ax$ under the assumption that $x$ is $k$-sparse.

(c) [Optional] Give an $n^{O(1)}$ time algorithm to recover $x$ from $Ax$ under the assumption that $x$ is $k$-sparse. You may choose specific values for the $\alpha^i_i$. Hint: look up syndrome decoding of Reed-Solomon codes.

2. In order to show that SSMP makes progress in each stage, we used a lemma that we will show in this problem.
Let \( x_1, \ldots, x_k \in \mathbb{R}^d \), and suppose that

\[
\sum_{i=1}^k \|x_i\|_1 \leq (1 + \delta) \|\sum_{i=1}^k x_i\|_1
\]

for some small enough \( \delta \) (say, \( \delta = 1/10 \)). In some sense, this is saying that there is not much “slack” in they are lined up head-to-tail.

(a) Let \( z = \sum_{i=1}^k x_i \). Show that \( \mathbb{E}_{i \in [k]} \|z - x_i\|_1 \leq (1 - \Omega(1)/k) \|z\|_1 \).

(b) Now suppose \( z = \sum_{i=1}^k x_i + w \) for some \( w \in \mathbb{R}^d \) with \( \|w\|_1 \leq \epsilon \|z\|_1 \) for small enough constant \( \epsilon \). Again, show that there exists an \( i \) such that \( \|z - x_i\|_1 \leq (1 - \Omega(1)/k) \|z\|_1 \).