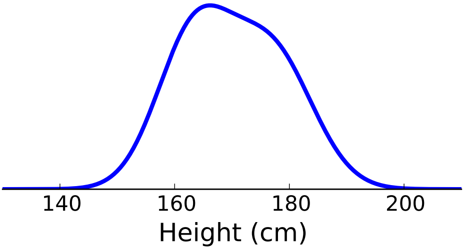
Tight Bounds for Learning a Mixture of Two Gaussians

Moritz Hardt **Eric Price**

Google Research UT Austin

2015-06-17

Problem



- Height distribution of American 20 year olds.

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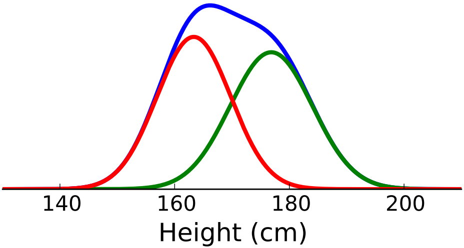

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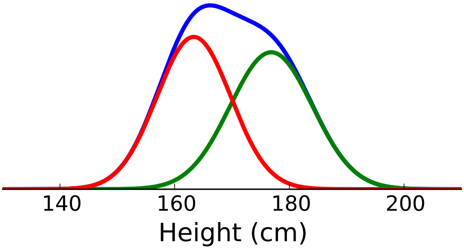

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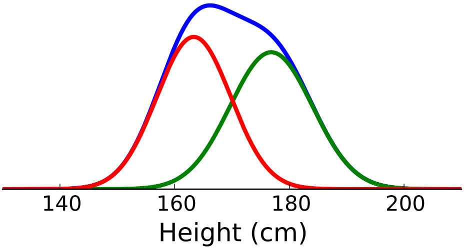

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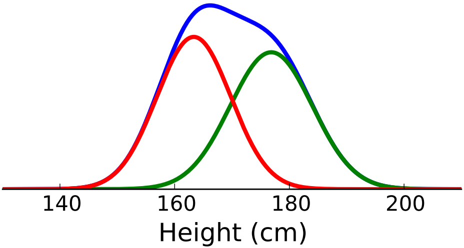

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- How many samples to learn μ_1, μ_2 to $\pm\epsilon\sigma$?
- *d*-dimensional setting: also learn weight, shoe size, ...

Gaussian Mixtures: Origins

III. *Contributions to the Mathematical Theory of Evolution.*

By KARL PEARSON, *University College, London.*

Communicated by Professor HENRICI, *F.R.S.*

Received October 18,—Read November 16, 1893.

[PLATES 1—5.]

CONTENTS.

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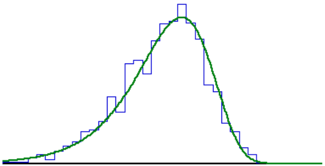

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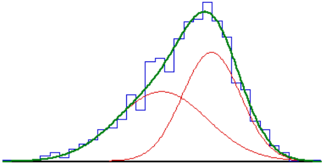

- Pearson's naturalist buddy measured lots of crab body parts.
- Most lengths seemed to follow the "normal" distribution (a recently coined name)
- But the "forehead" size wasn't symmetric.
- Maybe there were actually two species of crabs?

More previous work

- Pearson 1894: proposed method for 2 Gaussians

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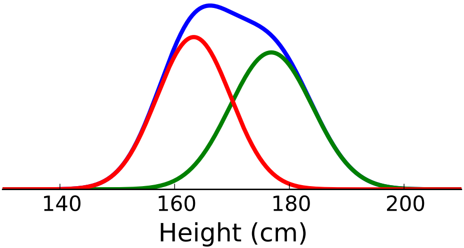
  - Lower bound extends to larger k.

Learning the components vs. learning the sum



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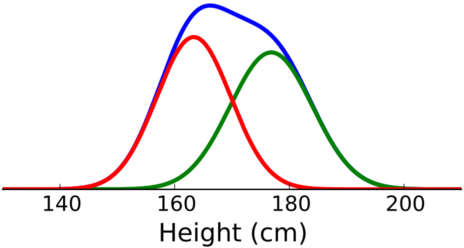

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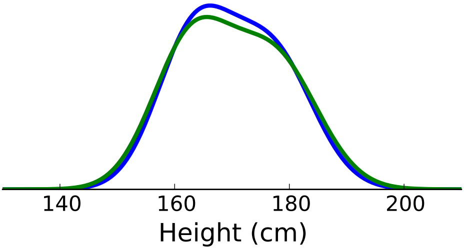

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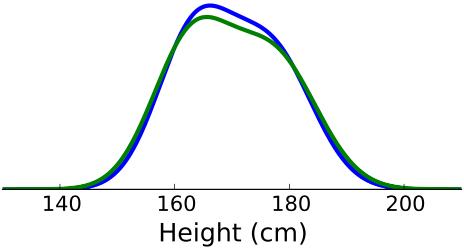

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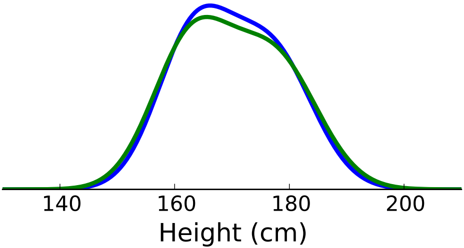

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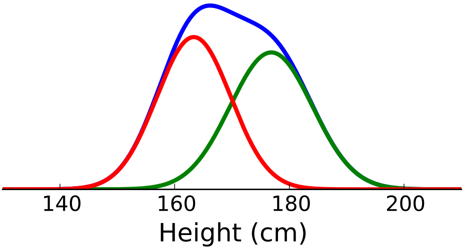

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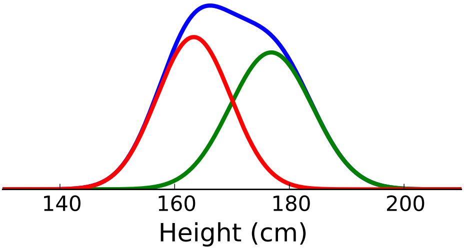

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  - Generic high-d TV estimation algs use 1d parameter estimation.

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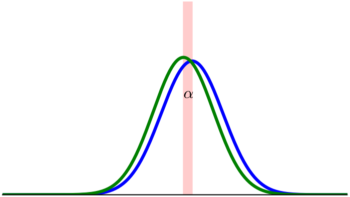

- More precisely: if two gaussians are α standard deviations apart, getting $\epsilon\alpha$ precision takes $\Theta(\frac{1}{\alpha^{12}\epsilon^2})$ samples.

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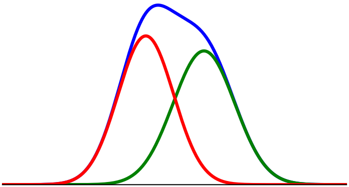

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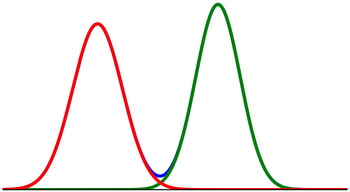

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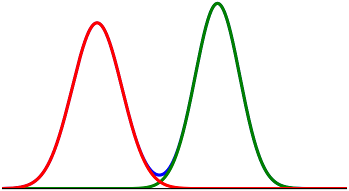

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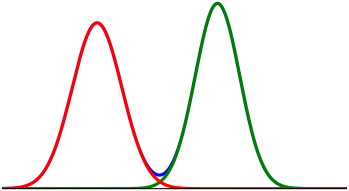

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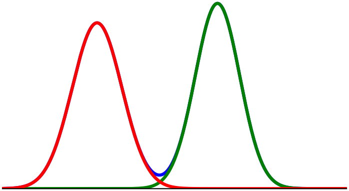

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- Caveat: assume p_1, p_2 are bounded away from zero throughout.

Outline

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1. Algorithm in One Dimension

2. Lower Bound

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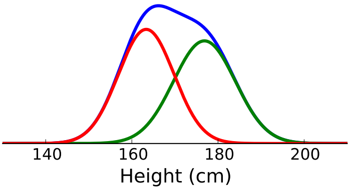
Method of Moments



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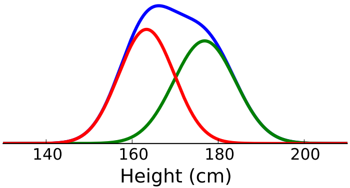

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\begin{aligned}
M_1 &:= \mathbb{E}[x^1] = p_1\mu_1 + p_2\mu_2 \\
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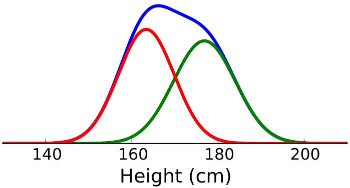

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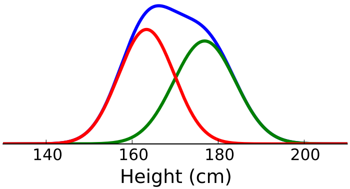

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- Use our samples to estimate the moments.
- Solve the system of equations to find the parameters.

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- Analogously, can assume $\min(\sigma_1, \sigma_2) = 0$ by converting to “excess moments”
  - $X_4 = M_4 - 3M_2^2$ is independent of adding $N(0, \sigma^2)$.

Method of Moments

Solving the system

Parameters	$\lambda > 0$ rate, or inverse scale
Support	$x \in [0, \infty)$
pdf	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$
Mean	λ^{-1}
Median	$\lambda^{-1} \ln(2)$
Mode	0
Variance	λ^{-2}
Skewness	2
Ex. kurtosis	6
Entropy	$1 - \ln(\lambda)$
MGF	$\left(1 - \frac{t}{\lambda}\right)^{-1}$ for $t < \lambda$
CF	$\left(1 - \frac{it}{\lambda}\right)^{-1}$
Fisher information	λ^{-2}

- Start with five parameters.
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  - Convert to "central moments"
  - $M_2' = M_2 - M_1^2$ is independent of translation.
- Analogously, can assume $\min(\sigma_1, \sigma_2) = 0$ by converting to "excess moments"
  - $X_4 = M_4 - 3M_2^2$ is independent of adding $N(0, \sigma^2)$.
  - "Excess kurtosis" coined by Pearson, appearing in every Wikipedia probability distribution infobox.

Method of Moments

Solving the system

Parameters	$\lambda > 0$ rate, or inverse scale
Support	$x \in [0, \infty)$
pdf	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$
Mean	λ^{-1}
Median	$\lambda^{-1} \ln(2)$
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- Leaves three free parameters.

Method of Moments: system of equations

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$$\alpha = -\mu_1\mu_2, \beta = \mu_1 + \mu_2, \gamma = \frac{\sigma_2^2 - \sigma_1^2}{\mu_2 - \mu_1}$$

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- Gives that

$$\begin{aligned}
X_3 &= \alpha(\beta + 3\gamma) \\
X_4 &= \alpha(-2\alpha + \beta^2 + 6\beta\gamma + 3\gamma^2) \\
X_5 &= \alpha(\beta^3 - 8\alpha\beta + 10\beta^2\gamma + 15\gamma^2\beta - 20\alpha\gamma) \\
X_6 &= \alpha(16\alpha^2 - 12\alpha\beta^2 - 60\alpha\beta\gamma + \beta^4 + 15\beta^3\gamma + 45\beta^2\gamma^2 + 15\beta\gamma^3)
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> *All my attempts to obtain a simpler set have failed... It is possible, however, that some other ... equations of a less complex kind may ultimately be found.*
>
> *—Karl Pearson*

Pearson's Polynomial

- Chug chug chug...

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- Get a 9th degree polynomial in the excess moments X_3, X_4, X_5:

$$
\begin{aligned}
p(\alpha) &= 8\alpha^9 + 28X_4\alpha^7 - 12X_3^2\alpha^6 + (24X_3X_5 + 30X_4^2)\alpha^5 \\
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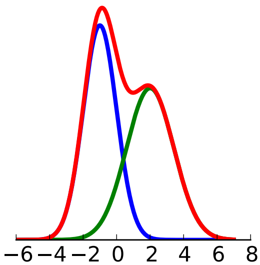
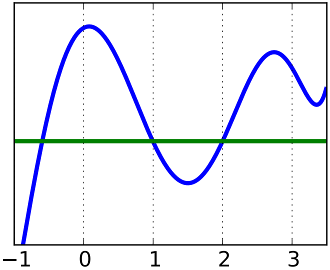
- Easy to go from solutions $\alpha = -\mu_1\mu_2$ to mixtures μ_i, σ_i, p_i.

Pearson's Polynomial





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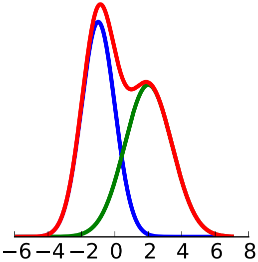

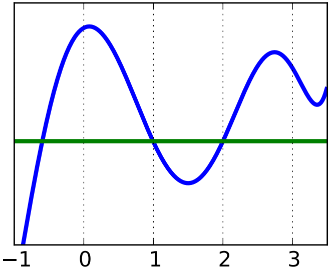

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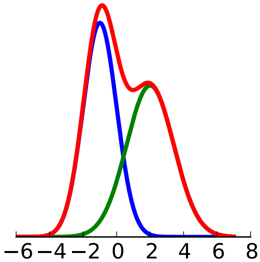

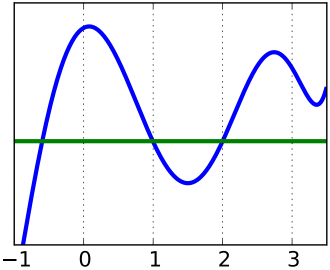

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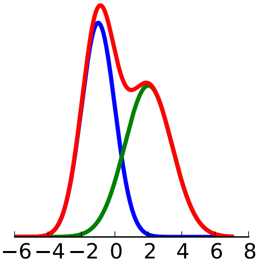

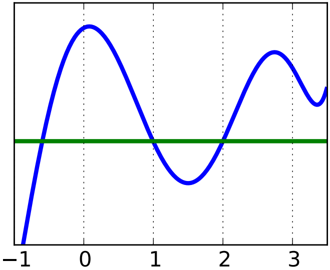

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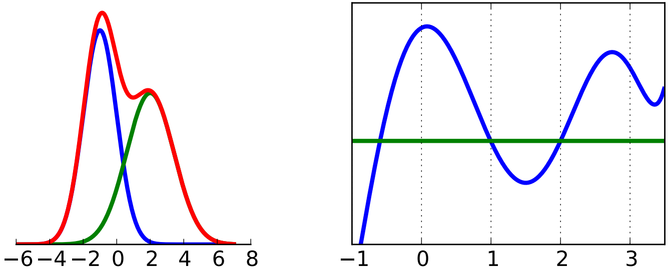

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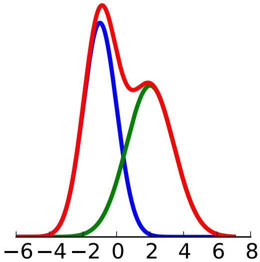

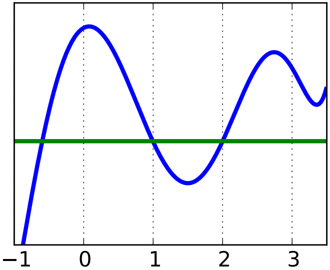

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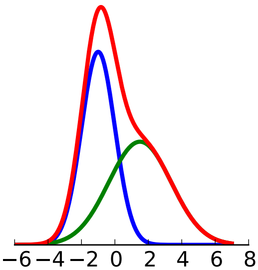

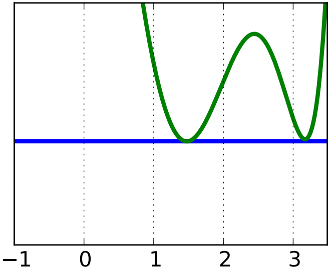

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  - Not when there's a double root.

Making it robust in all cases

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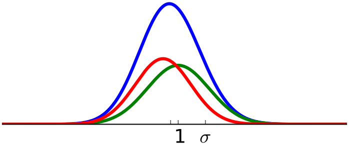
  - Getting α lets us estimate means, variances.

Result



- Scale so the excess moments are $O(1)$: μ_i are $\pm O(1)$.

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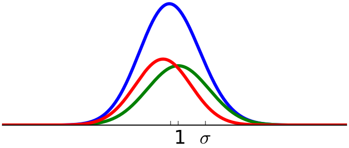

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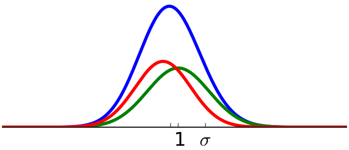

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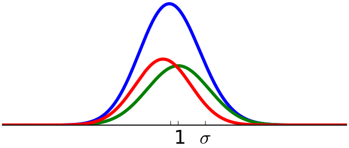

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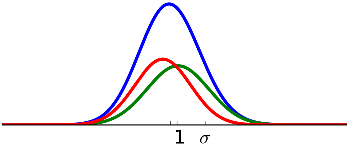

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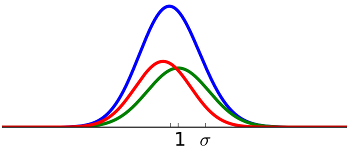

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  - In general, $O(1/\epsilon^{12})$ samples suffice to get $\epsilon\sigma$ accuracy.

Outline

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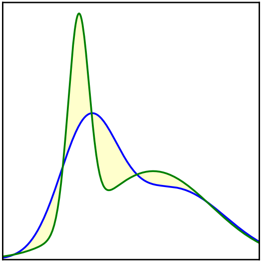

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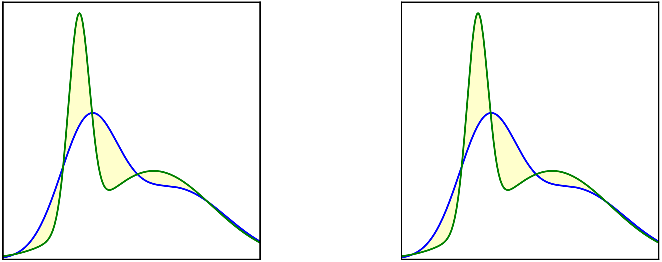

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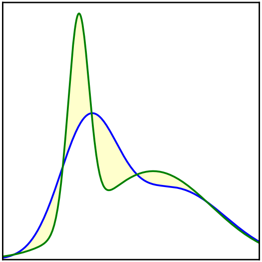

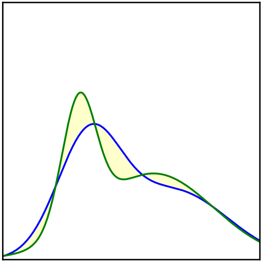

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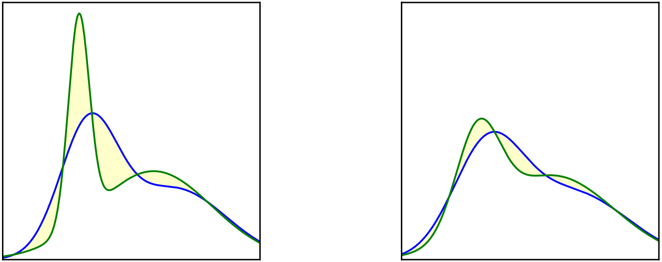

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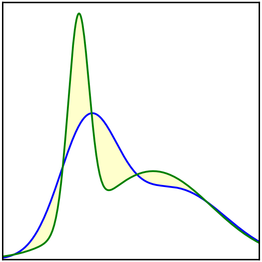

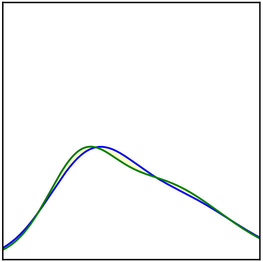

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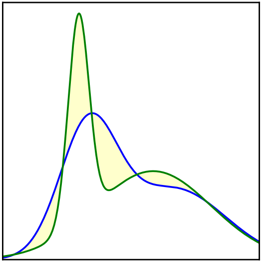

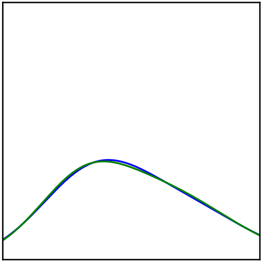

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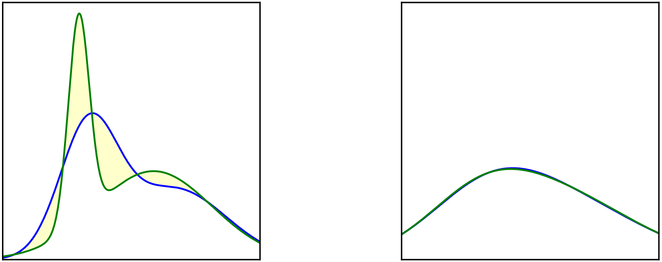

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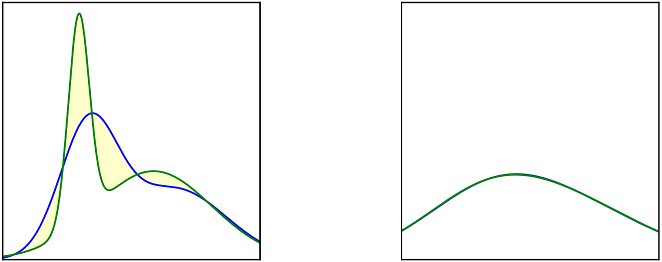

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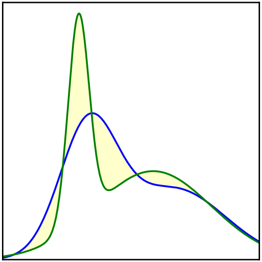

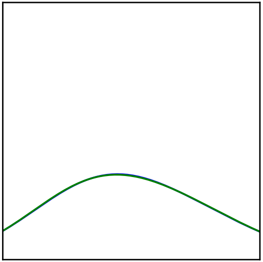

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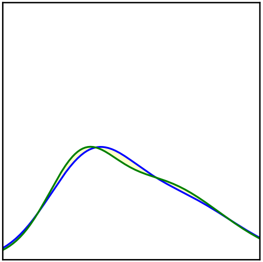
  - Add $N(0, \sigma^2)$ to each mixture for growing σ.
- Claim: $\Omega(\sigma^{12})$ samples necessary to distinguish the distributions.

Lower bound in one dimension



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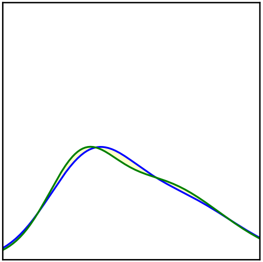

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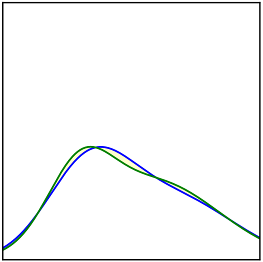

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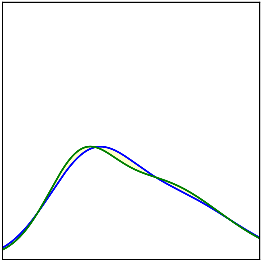

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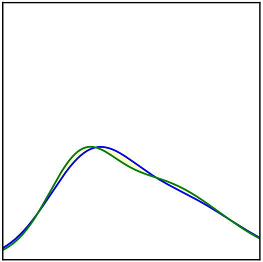

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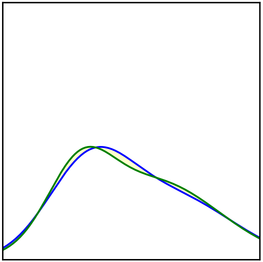

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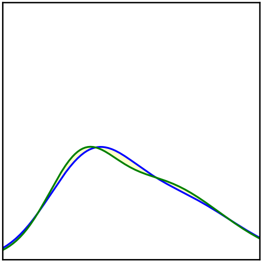

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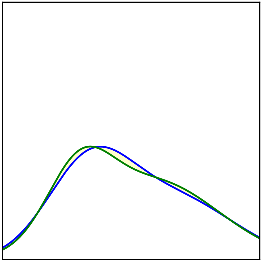

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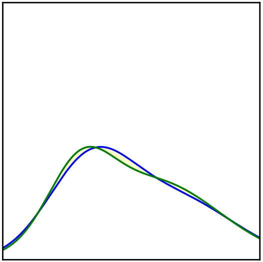

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  - $H^2 \lesssim TV \lesssim H$, but often $H \approx TV$.

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- Compare to $TV(p, q) = \frac{1}{2}\mathbb{E}_{x\sim p}[|\Delta(x)|]$

Bounding the Hellinger distance: our setting

Lemma

Let F, F' be two subgaussian distributions with k matching moments and constant parameters. Then for $G, G' = F + N(0, \sigma^2), F' + N(0, \sigma^2)$,

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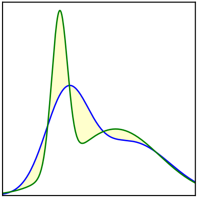
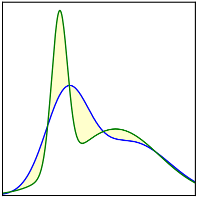
- Leaves $(1/\sigma^{k+1})^2$ as largest remaining term.

Lower bound in one dimension

- Add $N(0, \sigma^2)$ to two mixtures with five matching moments.





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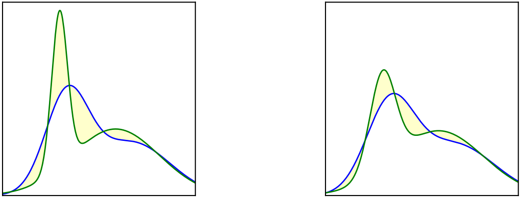

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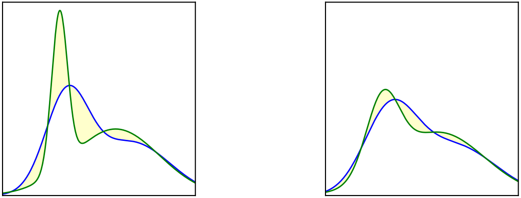

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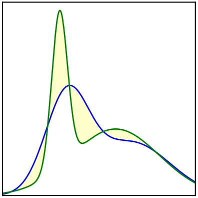

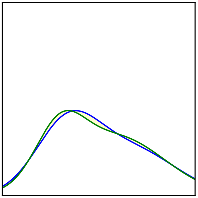

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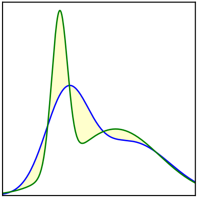

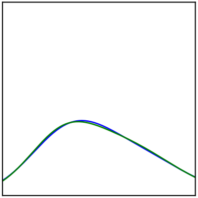

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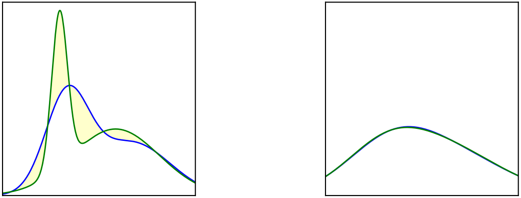

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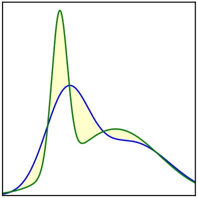

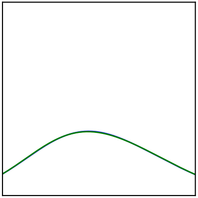

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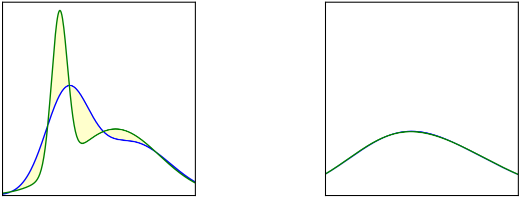

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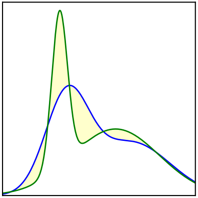

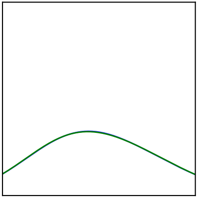

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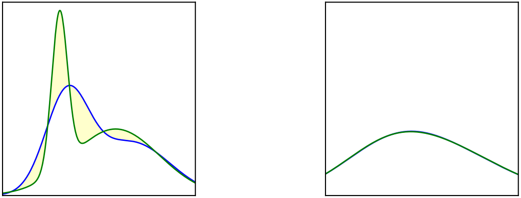

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Outline

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Algorithm in d dimensions

- Want to learn average male/female height, weight, shoe size, ...
  - (And covariance matrix)
- Look at individual attributes to get all these.
- Just need to know: is the taller group also heavier or lighter?
- Suffices to consider $d = 2$:
  - Does μ_i go with μ_j or μ_j'?
  - Project onto a random direction $e_i \sin\theta + e_j \cos\theta$.
  - (μ_i, μ_j) usually has a significantly different projection from (μ_i, μ_j').
- Thus we can piece them together by solving the $O(d^2)$ one dimensional problems.
- For covariances: reduce to $d = 4$, so $O(d^4)$ one dimensional problems.
- Only loss is $\log(1/\delta) \to \log(d/\delta)$:

$$\Theta(1/\epsilon^{12} \log(d/\delta)) \text{ samples}$$

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- TV estimation in d dimensions with d/ϵ^c rather than d^{30}/ϵ^c?