BLASFEO

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BLIS retreat
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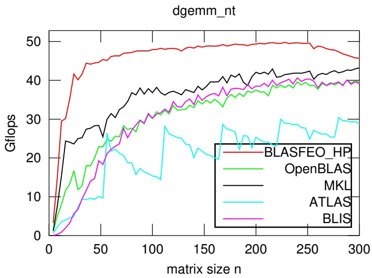
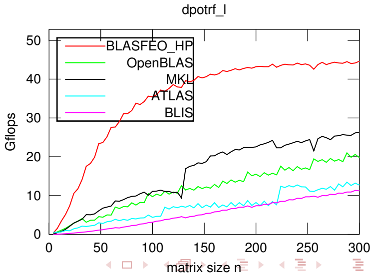
BLASFEO

- Basic Linear Algebra Subroutines For Embedded Optimization

BLASFEO performance

- Intel Core i7 4800MQ
  - BLASFEO HP
  - OpenBLAS 0.2.19
  - MKL 2017.2.174
  - ATLAS 3.10.3
  - BLIS 0.1.6
- CAVEAT: BLASFEO is not API compatible with BLAS & LAPACK



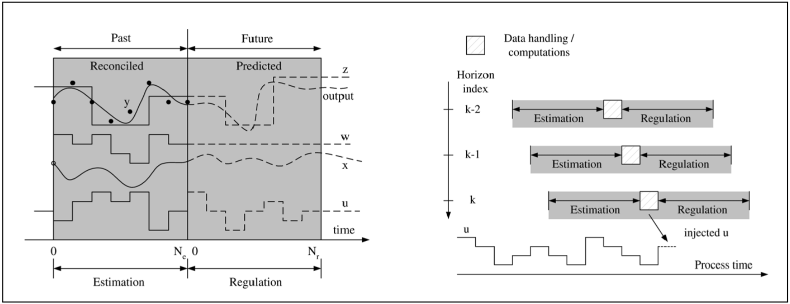


Framework: embedded optimization

- Framework: embedded optimization and control

Embedded control

Model Predictive Control (and Moving Horizon Estimation)



- solve an optimization problem on-line
- sampling times in milli- or micro-second range

Karush-Kuhn-Tucker optimality conditions

KKT system (for $N = 2$)

$$
\left[\begin{array}{ccc|ccc|c}
Q_0 & S_0^T & A_0^T & & & & \\
S_0 & R_0 & B_0^T & & & & \\
A_0 & B_0 & & -I & & & \\
\hline
 & & -I & Q_1 & S_1^T & A_1^T & \\
 & & & S_1 & R_1 & B_1 & \\
 & & & A_1 & B_1 & & -I \\
\hline
 & & & & & -I & Q_2
\end{array}\right]
\left[\begin{array}{c}
x_0 \\ u_0 \\ \lambda_0 \\ \hline x_1 \\ u_1 \\ \lambda_1 \\ \hline x_2
\end{array}\right]
=
\left[\begin{array}{c}
-q_0 \\ -r_0 \\ -b_0 \\ \hline -q_1 \\ -r_1 \\ -b_1 \\ \hline -q_2
\end{array}\right]
$$

- Large, structured system of linear equations
- Sub-matrices are assumed dense or diagonal

Framework: embedded optimization

Assumptions about embedded optimization:

- Computational speed is a key factor: solve optimization problems in real-time on resources-constrained hardware.
- Data matrices are reused several times (e.g. at each optimization algorithm iteration): look for a good data structure.
- Structure-exploiting algorithms can exploit the high-level sparsity pattern: data matrices assumed dense.
- Size of matrices is relatively small (tens or few hundreds): generally fitting in cache.
- Limitations of embedded optimization hardware and toolchain: no external libraries and (static/dynamic) memory allocation

The origin

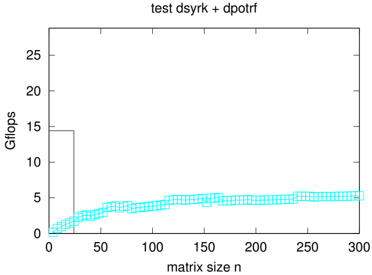
- HPMPC: library for High-Performance implementation of solvers for Model Predictive Contorl

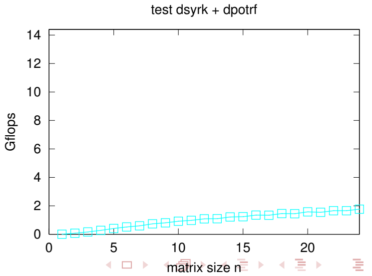
How to optimize syrk+potrf (for embedded optimization)

- How to optimize dsyrk + dpotrf (for embedded optimization)
- Test operation:

$$\mathcal{L} = \left(\mathcal{Q} + \mathcal{A} \cdot \mathcal{A}^T\right)^{1/2}$$

- NetlibBLAS

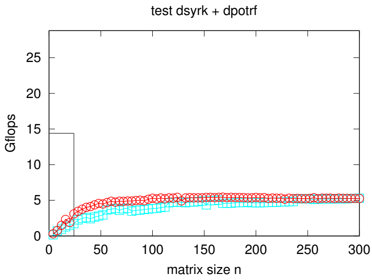


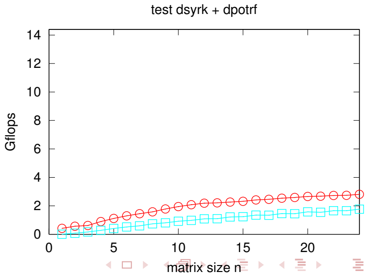


How to optimize syrk+potrf (for embedded optimization)

Code Generation

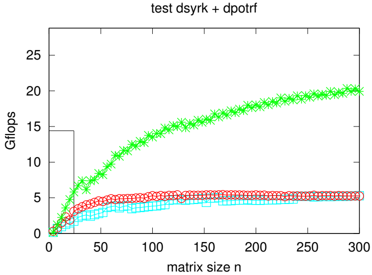
- e.g. fix the size of the loops: compiler can unroll loops and avoid branches
- need to generate the code for each problem size

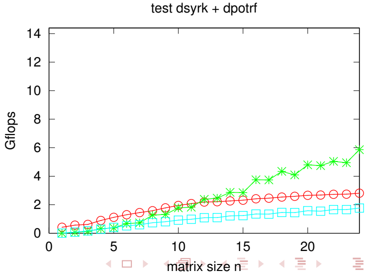




How to optimize syrk+potrf (for embedded optimization)

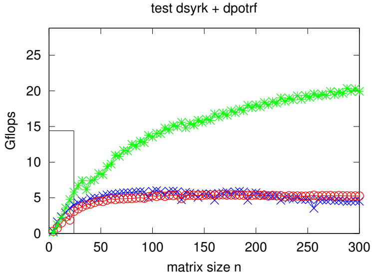
OpenBLAS

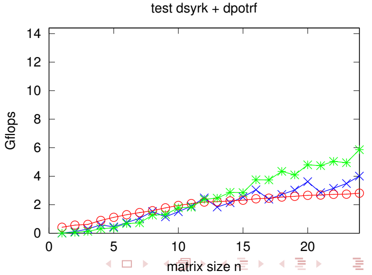




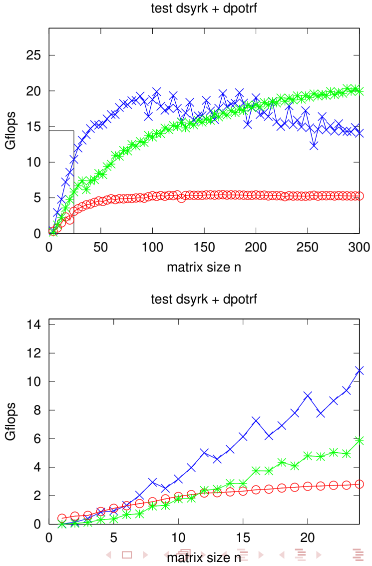
How to optimize syrk+potrf (for embedded optimization)

HPMPC - register blocking





How to optimize syrk+potrf (for embedded optimization)

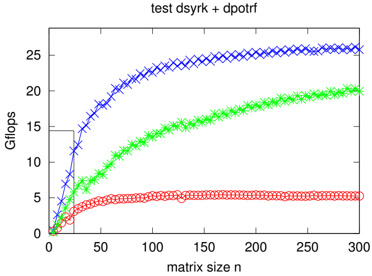


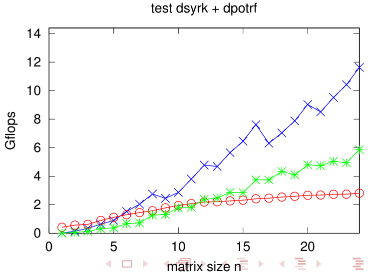
HPMPC - SIMD instructions

- performance drop for *n* multiple of 32 - limited cache associativity

HPMPC - panel-major matrix format

- panel-major matrix format
- smooth performance





Access pattern in optimized BLAS

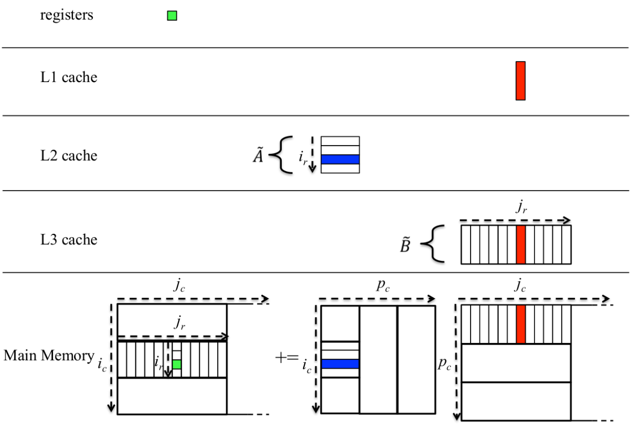
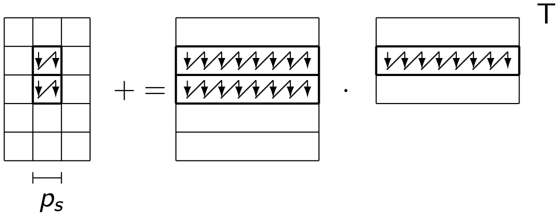


Figure: Access pattern of data in different cache levels for the `dgemm` routine in GotoBLAS/OpenBLAS/BLIS. Data is packed (on-line) into buffers following the access pattern.

Panel-major matrix format



- matrix format: can do any operation, also on sub-matrices
- matrix elements are stored in (almost) the same order such as the `gemm` kernel accesses them
- optimal 'NT' `gemm` variant (A not-transposed, B transposed)
- panels width p_s is the same for the left and the right matrix operand, as well as for the result matrix

Optimized BLAS vs HPMPC software stack

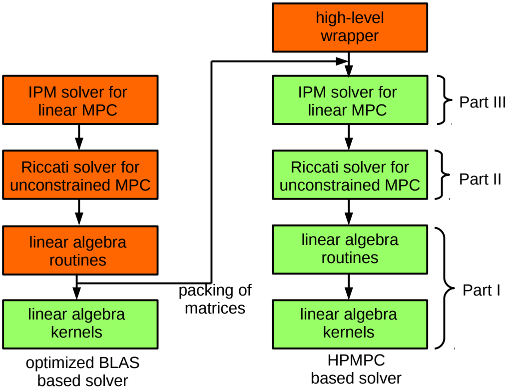
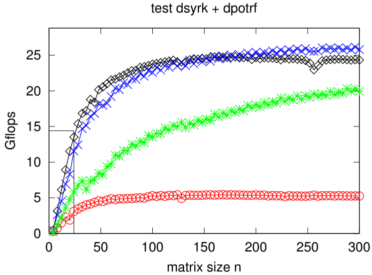


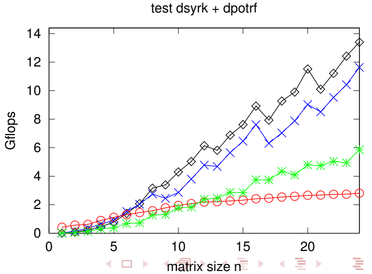
Figure: Structure of a Riccati-based IPM for linear MPC problems when implemented using linear algebra in either optimized BLAS or HPMPC. Routines in the orange boxes use matrices in column-major format, routines in the green boxes use matrices in panel-major format.

How to optimize syrk+potrf (for embedded optimization)

HPMPC - merging of linear algebra routines

- specialized kernels for complex operations
- improves small-scale performance
- worse large-scale performance





Merging of linear algebra routines: syrk + potrf

$$\mathcal{L} = \left(\mathcal{Q} + \mathcal{A}\cdot\mathcal{A}^T\right)^{1/2} =$$

$$\begin{bmatrix}\mathcal{L}_{00} & * & * \\ \mathcal{L}_{10} & \mathcal{L}_{11} & * \\ \mathcal{L}_{20} & \mathcal{L}_{21} & \mathcal{L}_{22}\end{bmatrix} = \left(\begin{bmatrix}\mathcal{Q}_{00} & * & * \\ \mathcal{Q}_{10} & \mathcal{Q}_{11} & * \\ \mathcal{Q}_{20} & \mathcal{Q}_{21} & \mathcal{Q}_{22}\end{bmatrix} + \begin{bmatrix}\mathcal{A}_0 \\ \mathcal{A}_1 \\ \mathcal{A}_2\end{bmatrix}\cdot\begin{bmatrix}\mathcal{A}_0^T & \mathcal{A}_1^T & \mathcal{A}_2^T\end{bmatrix}\right)^{1/2} =$$

$$\begin{bmatrix}(\mathcal{Q}_{00}+\mathcal{A}_0\cdot\mathcal{A}_0^T)^{1/2} & * & * \\ (\mathcal{Q}_{10}+\mathcal{A}_1\cdot\mathcal{A}_0^T)\mathcal{L}_{00}^{-T} & (\mathcal{Q}_{11}+\mathcal{A}_1\cdot\mathcal{A}_1^T-\mathcal{L}_{10}\cdot\mathcal{L}_{10}^T)^{1/2} & * \\ (\mathcal{Q}_{20}+\mathcal{A}_2\cdot\mathcal{A}_0^T)\mathcal{L}_{00}^{-T} & (\mathcal{Q}_{21}+\mathcal{A}_2\cdot\mathcal{A}_1^T-\mathcal{L}_{20}\cdot\mathcal{L}_{10}^T)\mathcal{L}_{11}^{-T} & (\mathcal{Q}_{22}+\mathcal{A}_2\cdot\mathcal{A}_2^T-\mathcal{L}_{20}\cdot\mathcal{L}_{20}^T-\mathcal{L}_{21}\cdot\mathcal{L}_{21}^T)^{1/2}\end{bmatrix}$$

- each sub-matrix computed using a single specialized kernel
  - reduce number of function calls
  - reduce number of load and store of the same data

The present

- BLASFEO

BLASFEO

- aim: satisfy the need for high-performance linear algebra routines for small dense matrices
- mean: adapt high-performance computing techniques to embedded optimization framework
- keep:
  - LA kernels (register-blocking, SIMD)
  - optimized data layout
- drop:
  - cache-blocking
  - on-line data packing
  - multi-thread (at least for now)
- add:
  - optimized (panel-major) matrix format
  - off-line data packing
  - assembly subroutines: tailored LA kernels & code reuse

Linear algebra level

- LA level definition
  - level 1: $\mathcal{O}(n)$ storage, $\mathcal{O}(n)$ flops
  - level 2: $\mathcal{O}(n^2)$ storage, $\mathcal{O}(n^2)$ flops
  - level 3: $\mathcal{O}(n^2)$ storage, $\mathcal{O}(n^3)$ flops
- in level 1 and level 2
  - reuse factor $\mathcal{O}(1)$
  - memory-bounded
- in level 3
  - reuse factor $\mathcal{O}(n)$
  - compute-bounded for large n
    - disregard $\mathcal{O}(n^2)$ terms
  - typically memory-bounded for small n
    - minimize $\mathcal{O}(n^2)$ terms $\Rightarrow$ BLASFEO playground!

BLASFEO RF

BLASFEO ReFerence

- 2x2 blocking for registers
- column-major matrix format
- small code size
- ANSI C code, no external dependencies
- good performance for tiny matrices

BLASFEO High-Performance

- optimal blocking for registers + vectorization
- no blocking for cache
- panel-major matrix format
- hand-written assembly kernels
- optimized for highest performance for matrices fitting in cache

BLASFEO WR

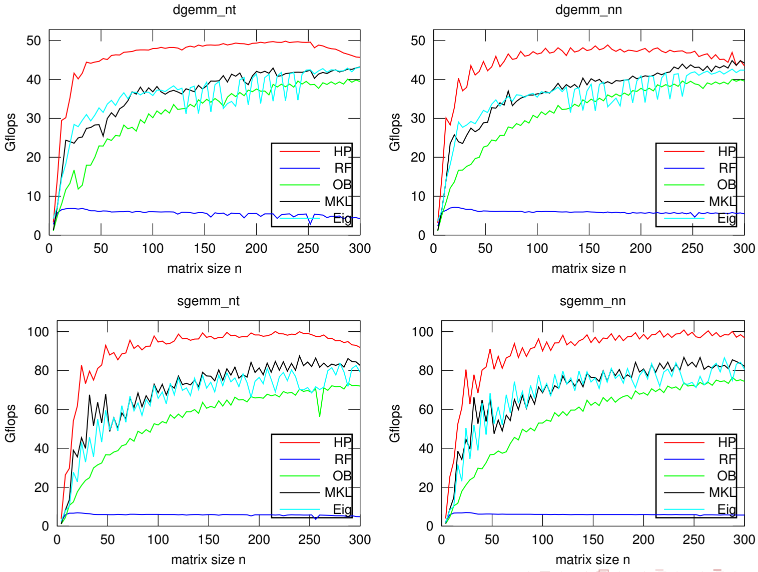
BLASFEO WRapper to BLAS and LAPACK

- provides a performance basis
- column-major matrix format
- optimized for many architectures
- possibly multi-threaded
- good performance for large matrices

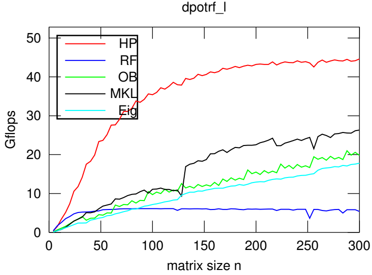
Matrix structure

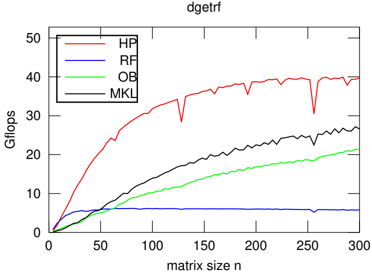
- problem: how to switch between panel-major and column-major formats?
- solution: use a C struct for the matrix "object"
  `(..., struct d_strmat *sA, int ai, int aj, ...)`
- if column-major, the first matrix element is at
  ```
  int lda = sA->m;
  double *ptr = sA->pA + ai + aj*lda;
  ```
- if panel-major, the first matrix element is at
  ```
  int ps = sA->ps;
  int sda = sA->cn;
  int air = ai&(ps-1);
  double *ptr = sA->pA + (ai-air)*sda + air + aj*bs;
  ```
- custom matrix struct allows other tricks
  - extra linear storage for inverse of diagonal in factorizations

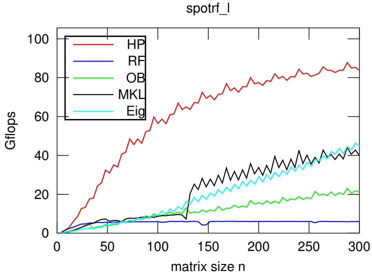
Some tests on Intel Core i7 4800MQ (Haswell)

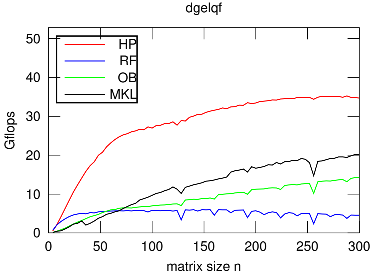


Some tests on Intel Core i7 4800MQ (Haswell)

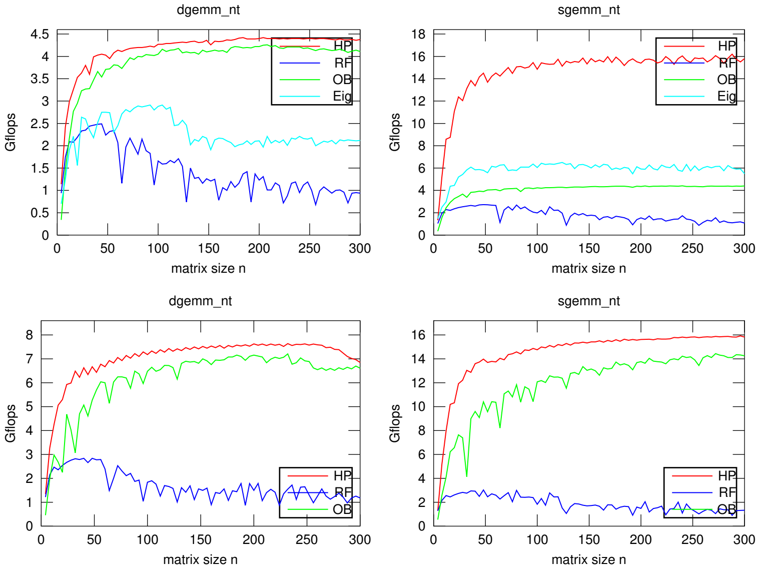




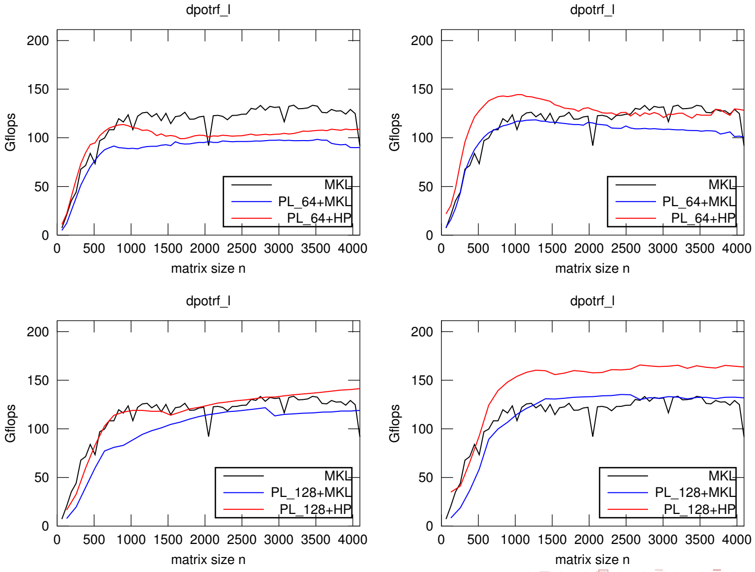




Some tests on ARM Cortex A15 and ARM Cortex A57



BLASFEO + PLASMA on Intel Core i7 4800MQ



The end (for now)

- thank you for your attention!
- questions?