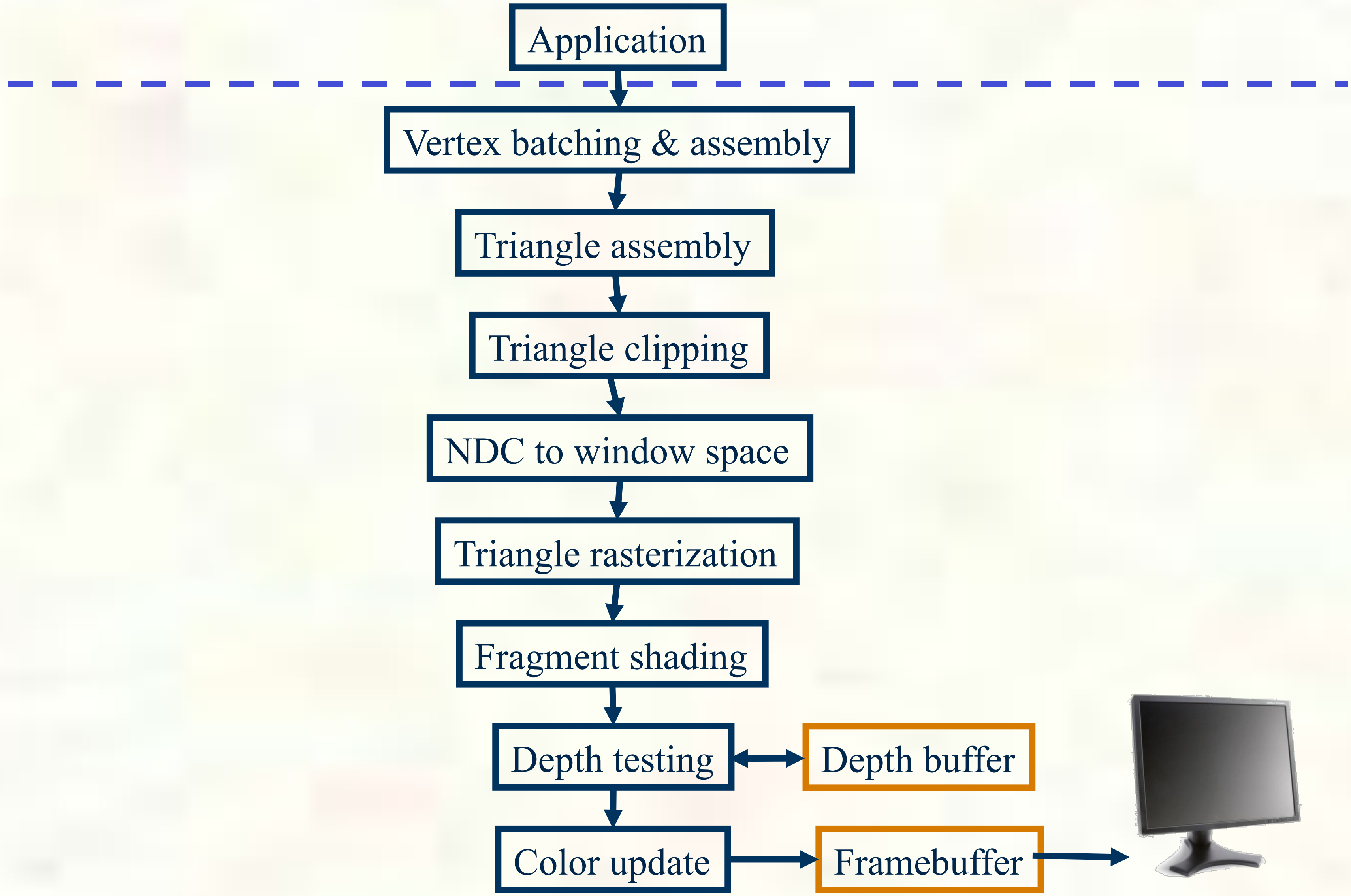
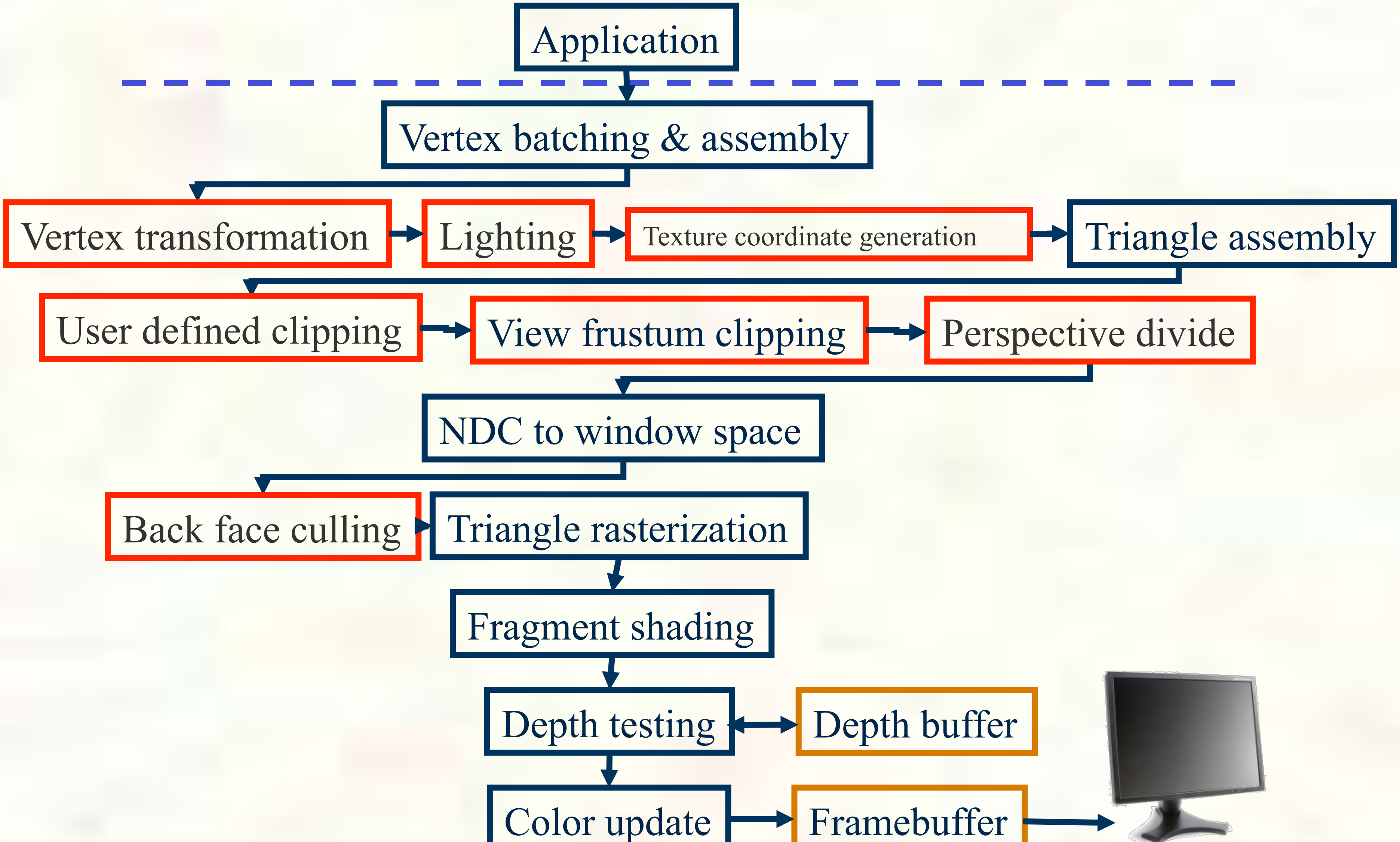
Viewing and Projections

Don Fussell
Computer Science Department
The University of Texas at Austin

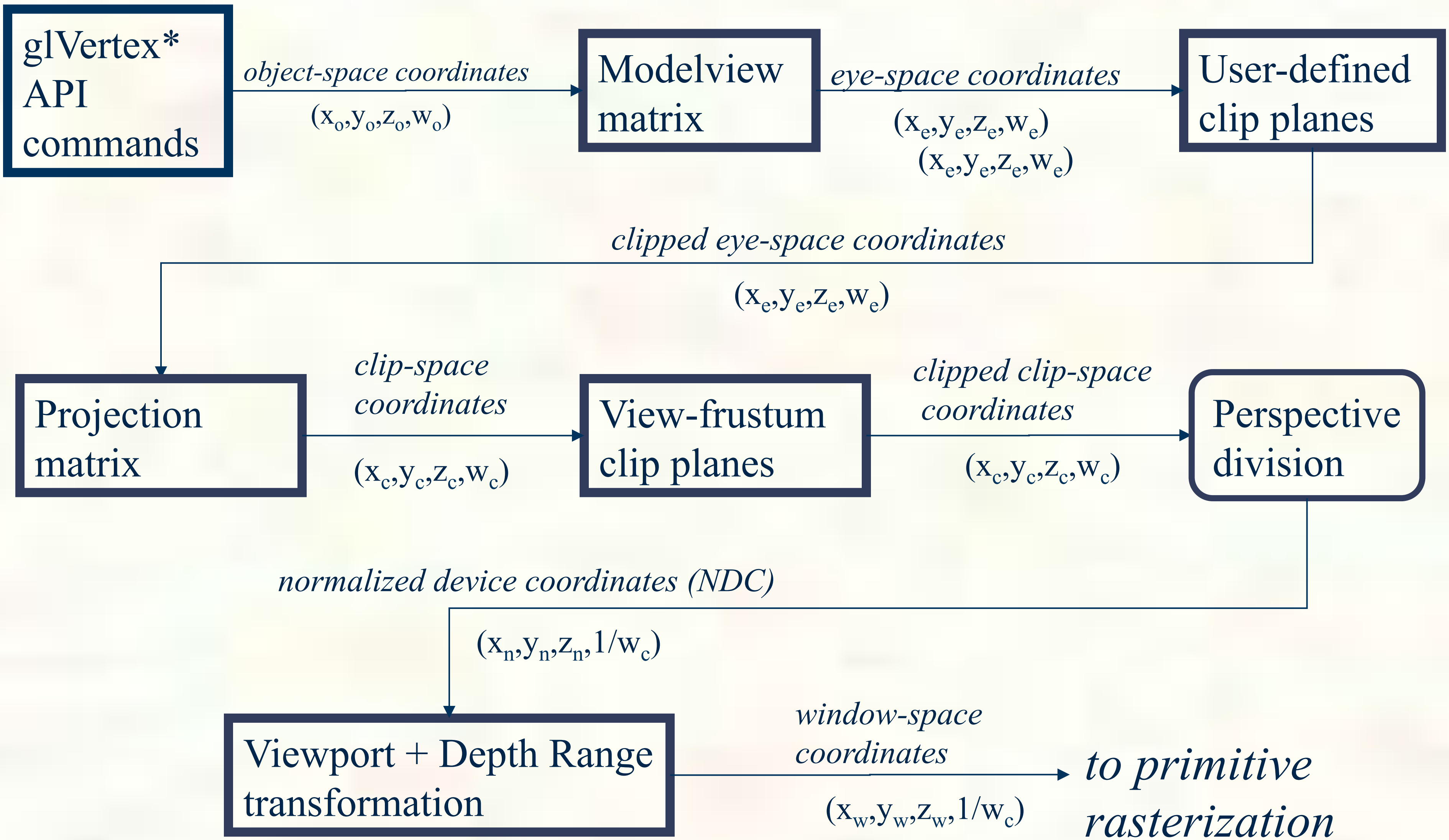
A Simplified Graphics Pipeline



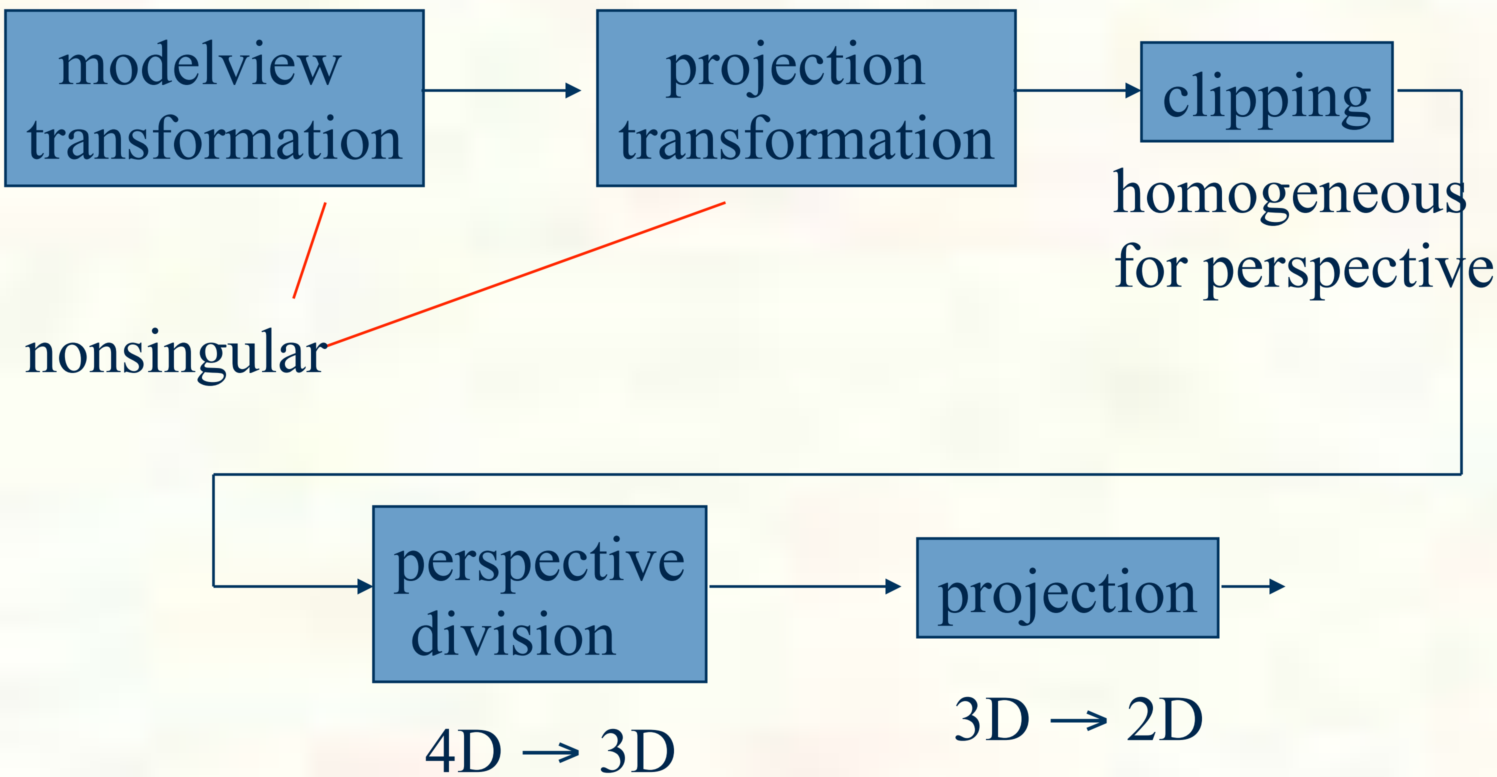
A few more steps expanded



Conceptual Vertex Transformation

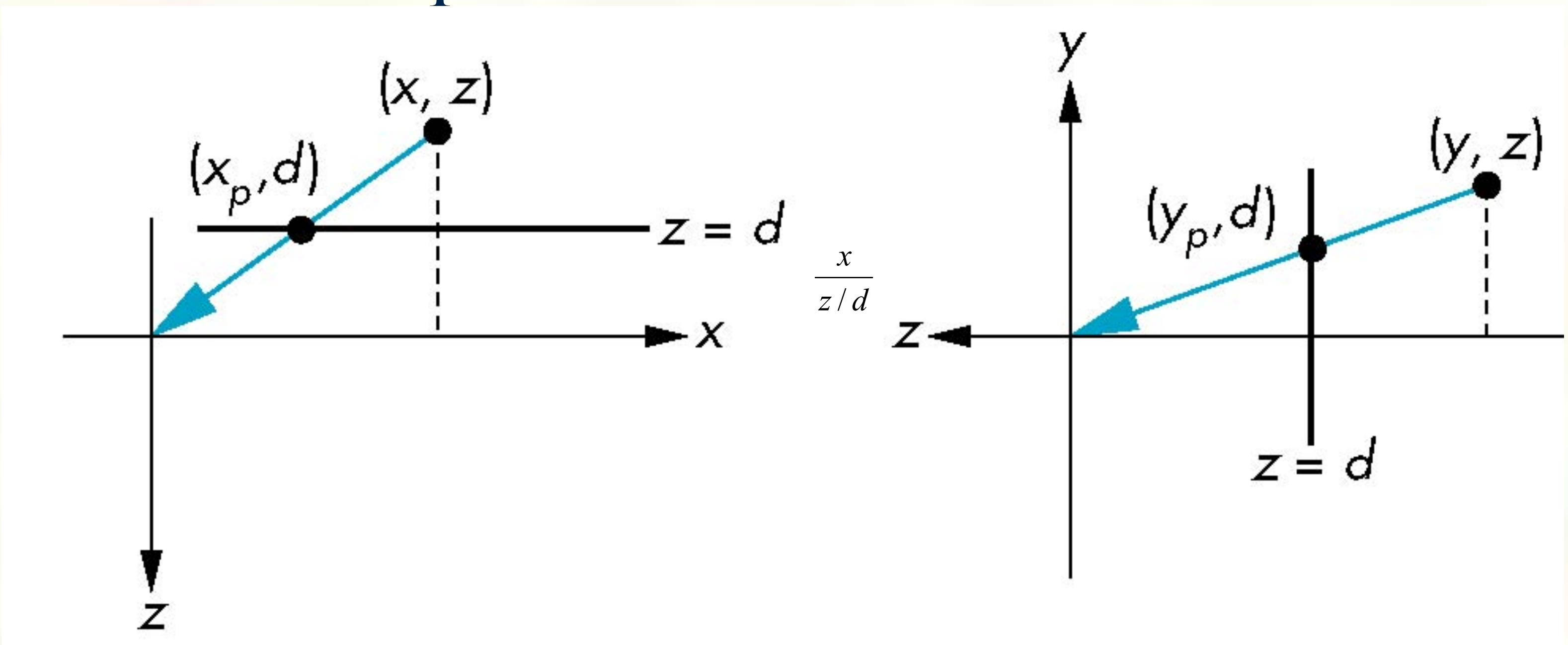


Pipeline View



Perspective Equations

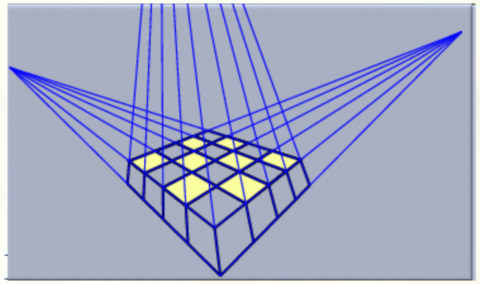
Consider top and side views



$$x_p = \frac{x}{z/d} \qquad y_p = \frac{y}{z/d} \qquad z_p = d$$

Four-component positions!

- Conventional geometry represents 3D points at (x,y,z) positions
  - Affine 3D positions, Cartesian coordinates
- Projective position concept
  - Use fourth coordinate: W
    - So (x,y,z,w)
  - (x/w, y/w, z/w) is the corresponding affine 3D position
  - Known as "homogeneous coordinates"
- Advantages
  - Represents perspective cleanly
  - Allows rasterization of external triangles
  - Puts off (expensive) division



Example, All Identical Positions

- Affine 3D
  - (x,y,z)
- Projective 3D
  - (x,y,z,w) → (x/w,y/w,z/w)

(1,-2.5,5,0.5)

(2,-5,10) → (2,-5,10,1)

(4,-10,20,2)

(-2,5,-10,-1)

Homogeneous Form

consider $\mathbf{q} = \mathbf{Mp}$ where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix}$$

Perspective Division

- However $w \neq 1$, so we must divide by w to return from homogeneous coordinates
- This *perspective division* yields

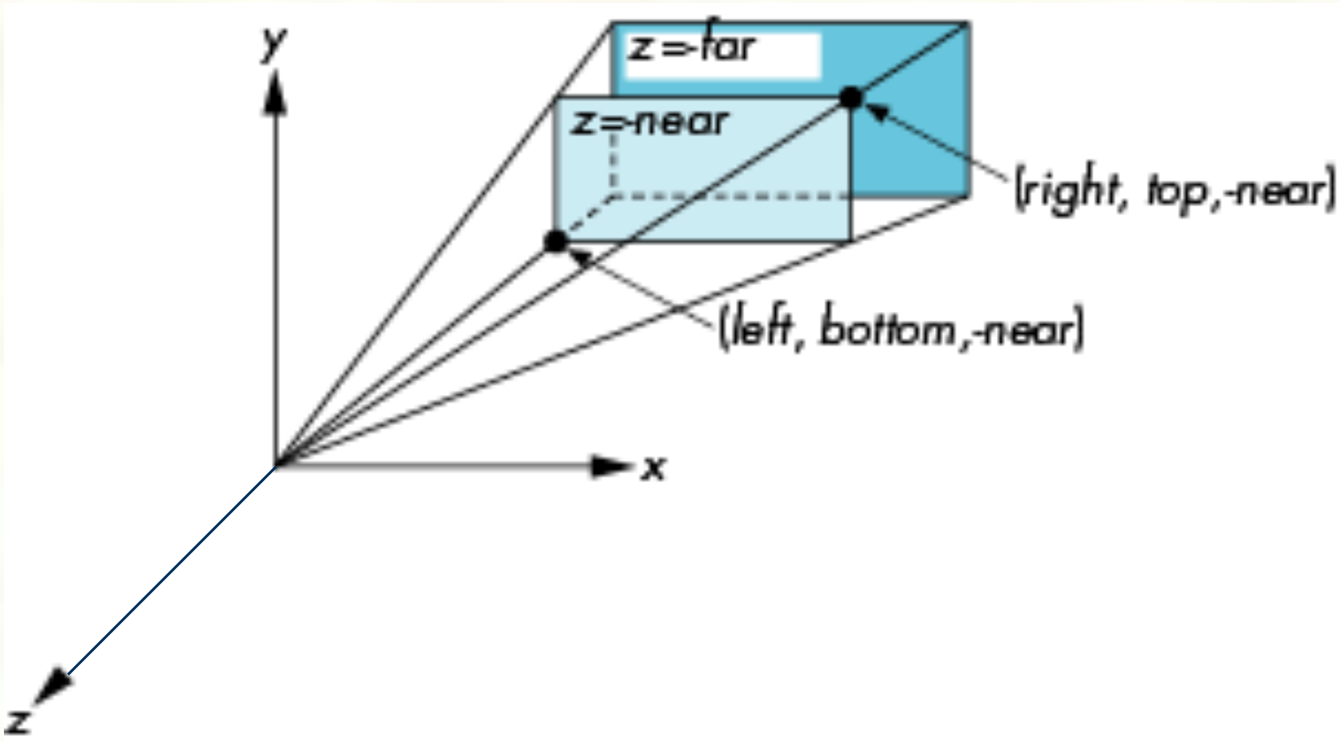
$$x_p = \frac{x}{z/d} \qquad y_p = \frac{y}{z/d} \qquad z_p = d$$

the desired perspective equations

- We will consider the corresponding clipping volume with the OpenGL functions

OpenGL Perspective

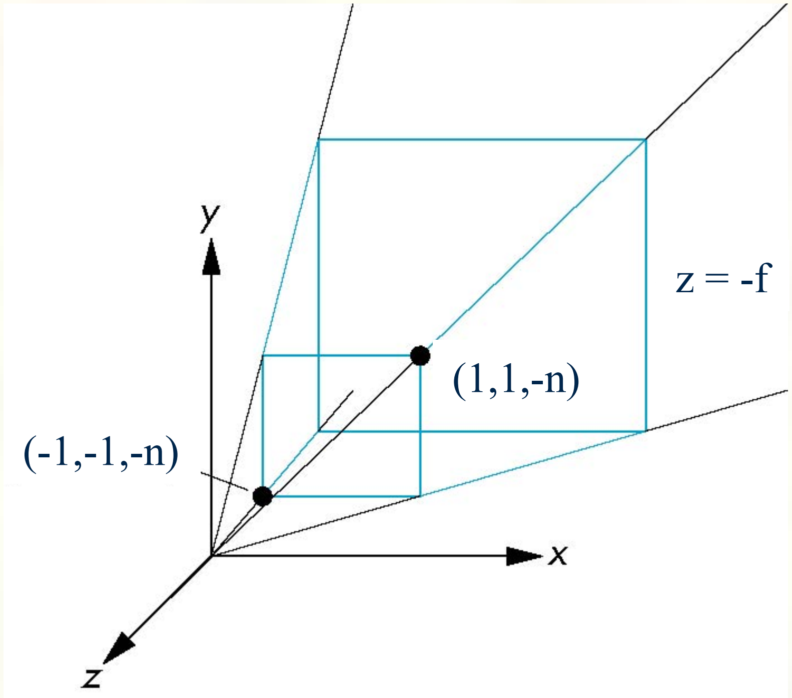
```
glFrustum(left,right,bottom,top,near,far
  )
```



Simple Perspective

Consider a simple perspective with the COP at the origin, the near clipping plane at z = -near, and a 90 degree field of view determined by the planes

$x = \pm z, y = \pm z$



Generalization

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

after perspective division, the point (x, y, z, 1) goes to

$x' = x/z$
$y' = y/z$
$z' = -(\alpha+\beta/z)$

which projects orthogonally to the desired point regardless of α and β

Picking α and β

If we pick

$$\alpha = -\frac{f+n}{f-n}$$

$$\beta = -\frac{2nf}{f-n}$$

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2nf}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

the near plane is mapped to $z = -1$
the far plane is mapped to $z = 1$
and the sides are mapped to $x = \pm 1$, $y = \pm 1$

Hence the new clipping volume is the default clipping volume

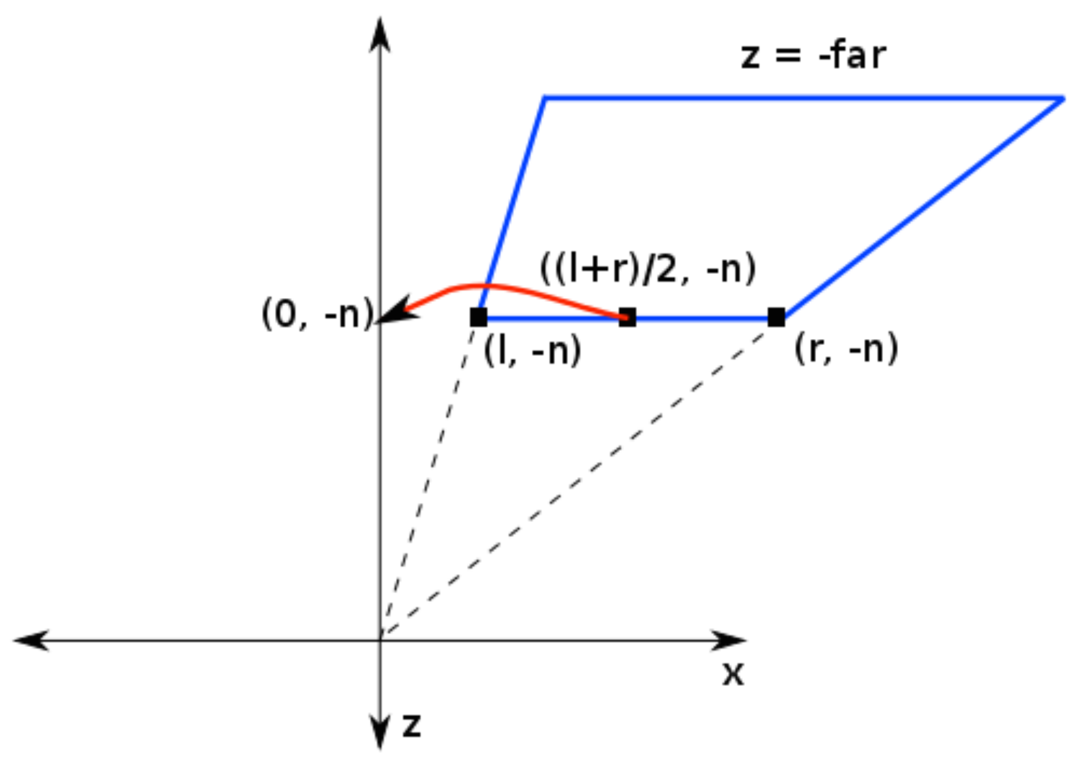
General Perspective Frustum

$$x' = x + \frac{l+r}{2n} z$$

$$y' = y + \frac{t+b}{2n} z$$

$$z' = z$$

$$H = \begin{bmatrix} 1 & 0 & \frac{l+r}{2n} & 0 \\ 0 & 1 & \frac{t+b}{2n} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Step 1: Shear to center on –z axis

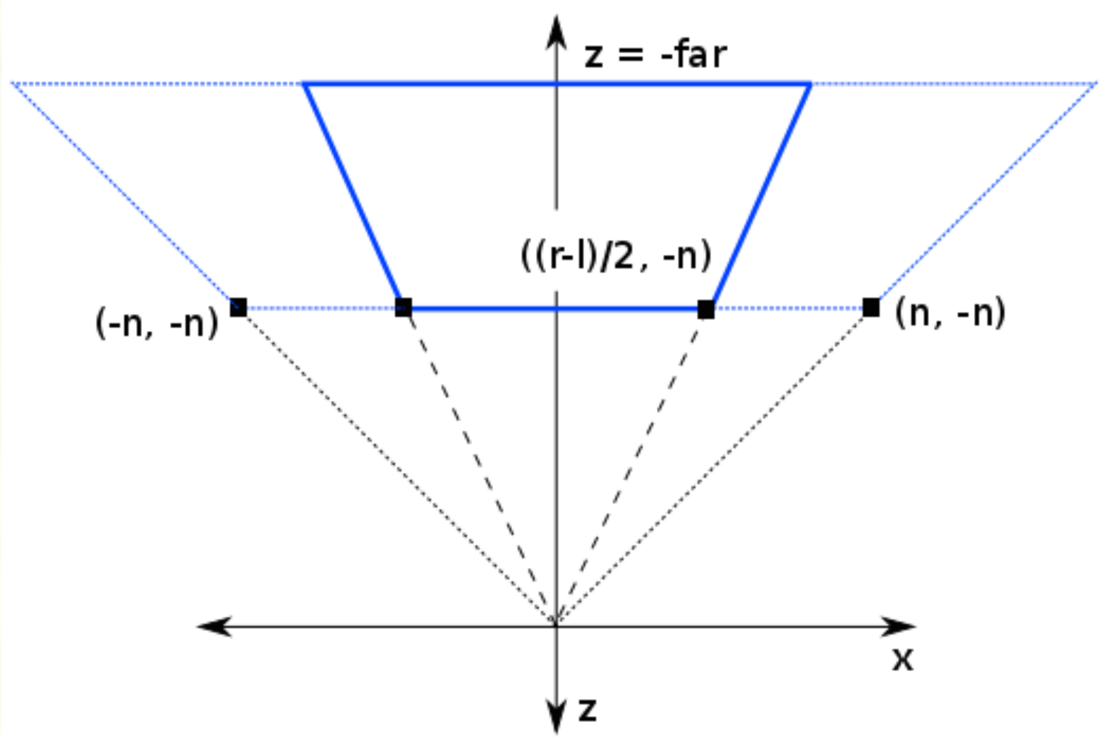
General Perspective Frustum

$$x' = \frac{2n}{r-l}x$$

$$y' = \frac{2n}{t-b}y$$

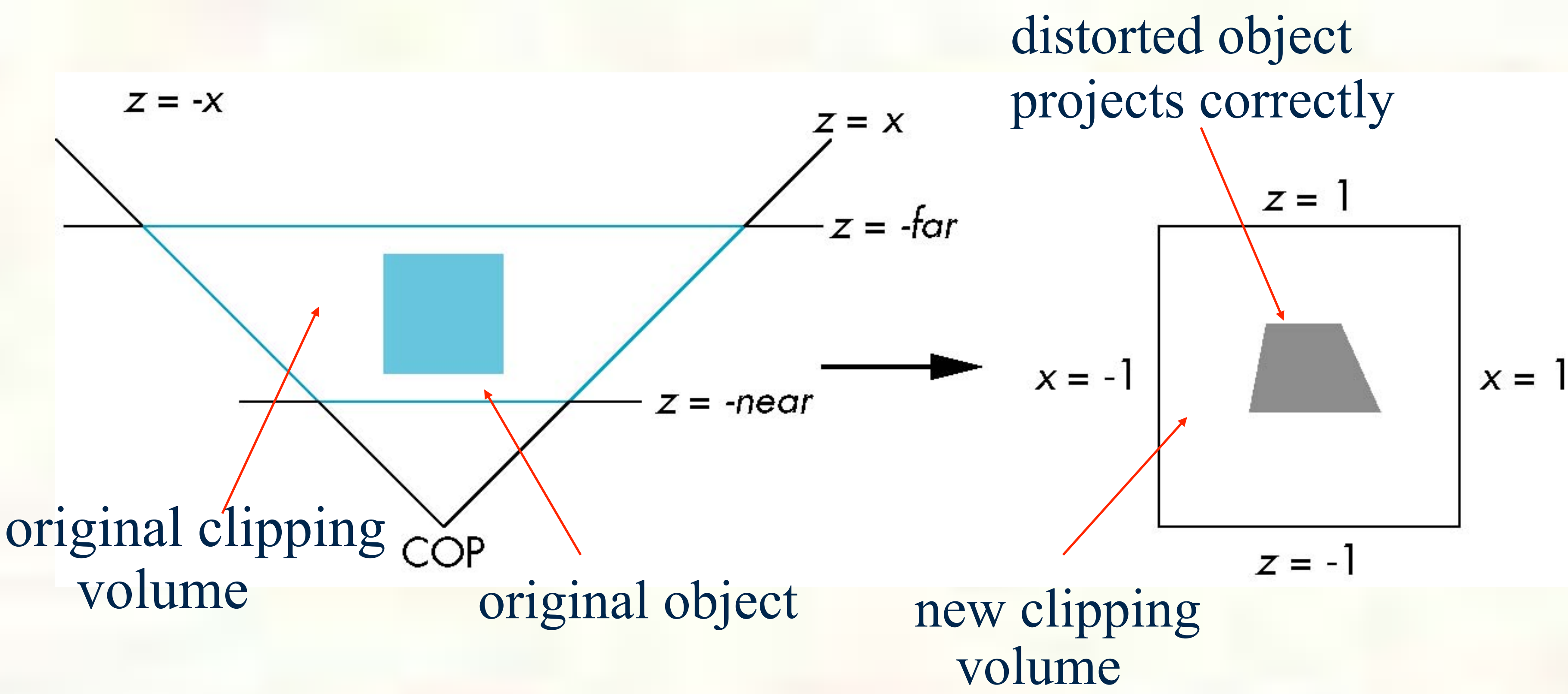
$$z' = z$$

$$S = \begin{bmatrix} \frac{2n}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2n}{t-b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Step 2: Scale so boundary slopes are ± 1

Normalization Transformation



OpenGL Perspective Matrix

- The normalization in **`glFrustum`** requires an initial shear to form a right viewing pyramid, followed by a scaling to get the normalized perspective volume. Finally, the perspective matrix results in needing only a final orthogonal transformation

$$\mathbf{P} = \mathbf{NSH}$$

our previously defined perspective matrix

shear and scale

Normalization

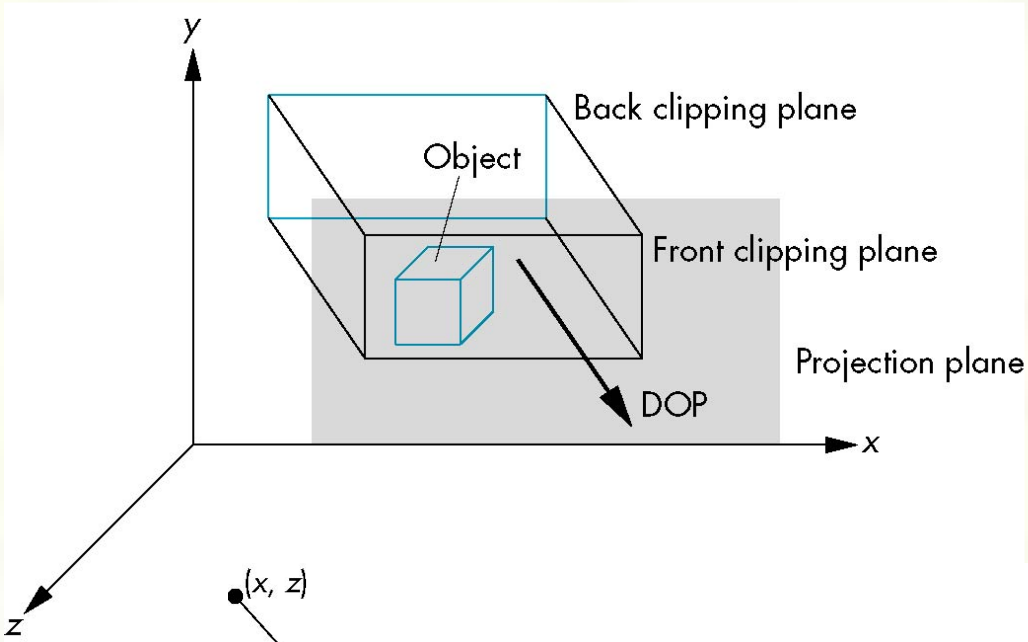
- Rather than derive a different projection matrix for each type of projection, we can convert all projections to orthogonal projections with the default view volume
- This strategy allows us to use standard transformations in the pipeline and makes for efficient clipping

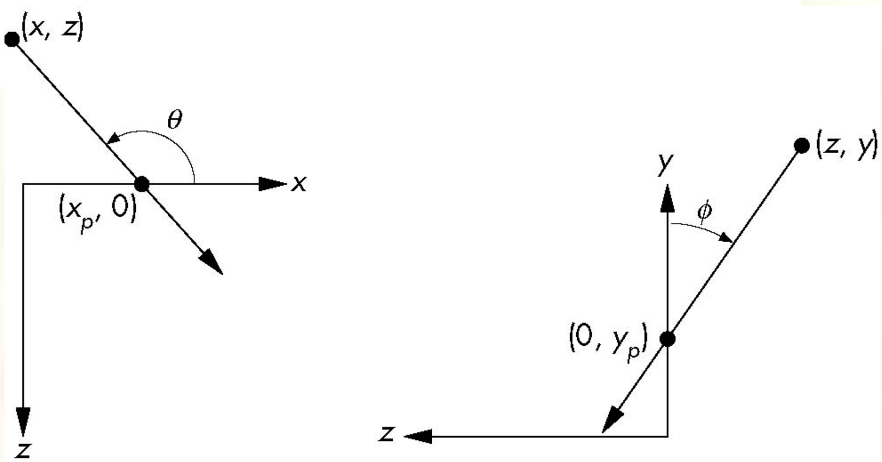
Oblique Projections

- The OpenGL projection functions cannot produce general parallel projections such as

- However if we look at the example of the cube it appears that the cube has been sheared
- Oblique Projection = Shear + Orthogonal Projection

General Shear





top view

side view

Shear Matrix

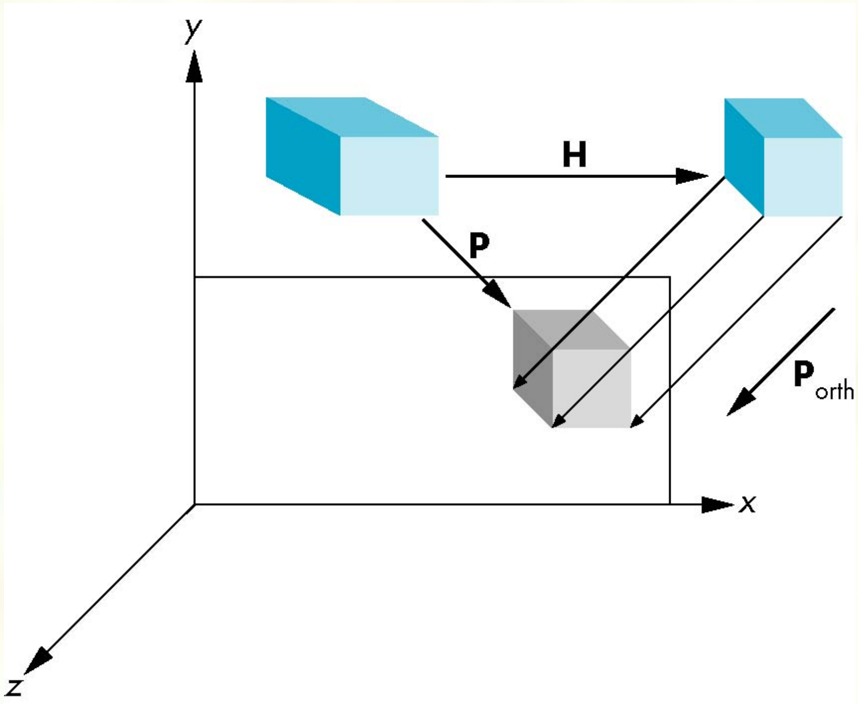
xy shear (*z* values unchanged)

$$\mathbf{H}(\theta,\phi) = \begin{bmatrix} 1 & 0 & -\cot\theta & 0 \\ 0 & 1 & -\cot\phi & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection matrix $\mathbf{P} = \mathbf{M}_{\text{orth}}\,\mathbf{H}(\theta,\phi)$

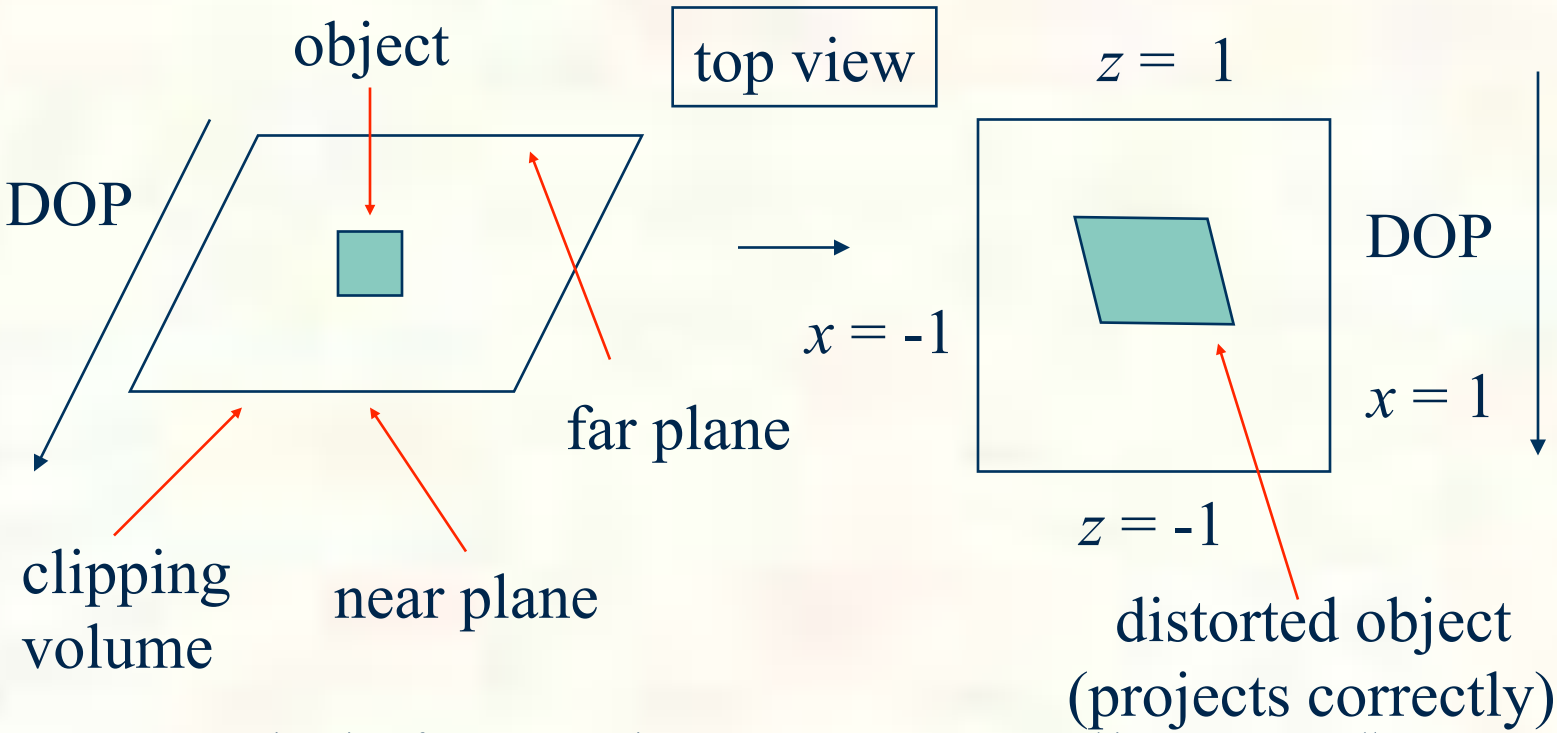
General case: $\mathbf{P} = \mathbf{M}_{\text{orth}}\,\mathbf{STH}(\theta,\phi)$

Equivalency



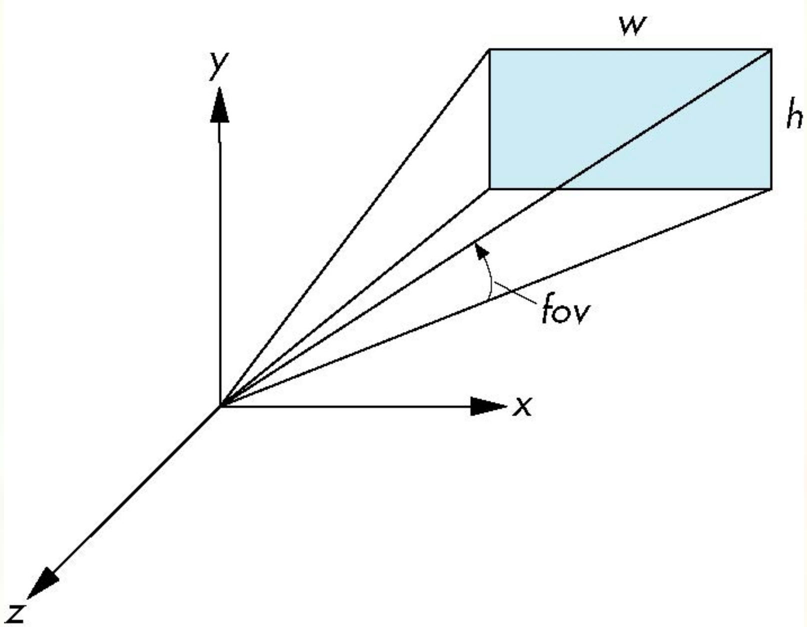
Effect on Clipping

- The projection matrix **P** = **STH** transforms the original clipping volume to the default clipping volume



Using Field of View

- With **`glFrustum`** it is often difficult to get the desired view
- **`gluPerpective(fovy, aspect, near, far)`** often provides a better interface

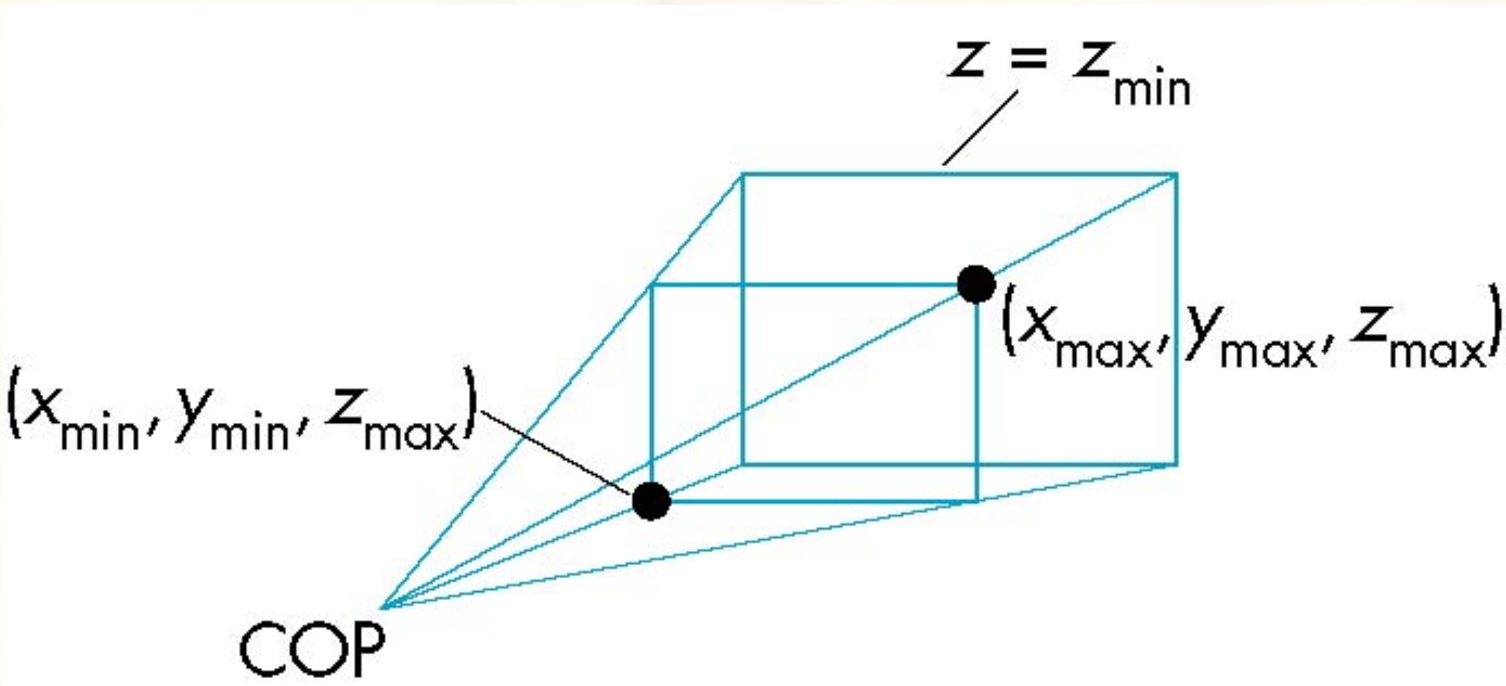


front plane

`aspect = w/h`

OpenGL Perspective

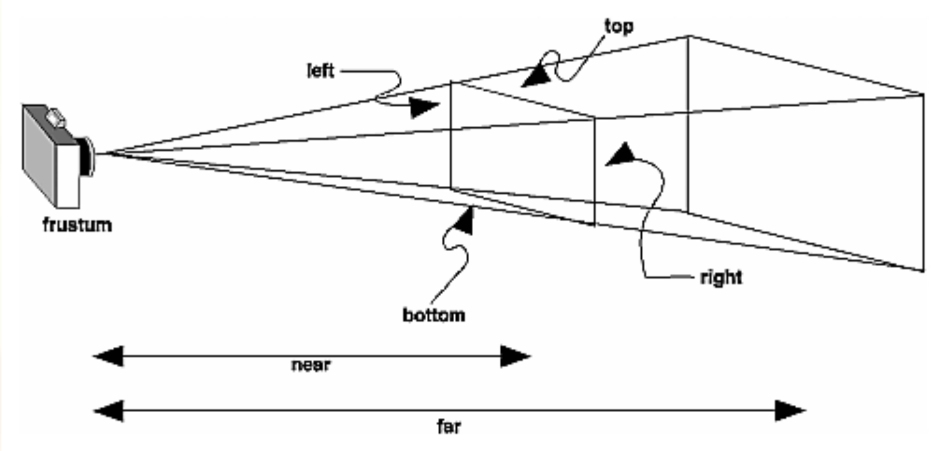
- **`glFrustum`** allows for an unsymmetric viewing frustum (although **`gluPerspective`** does not)



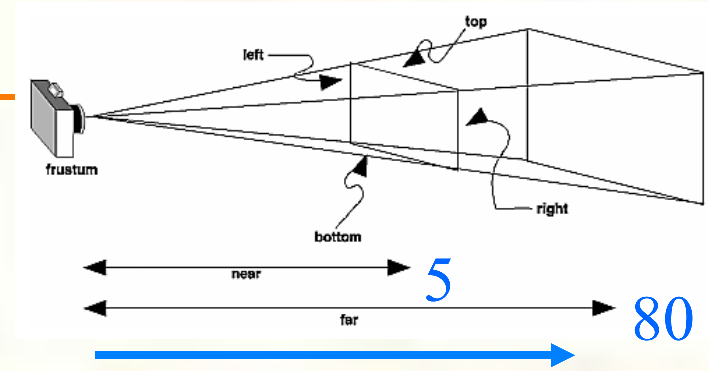
Frustum Transform

- Prototype
  - glFrustum(GLdouble left, GLdouble right, GLdouble bottom, GLdouble top, GLdouble near, GLdouble far)
- Post-concatenates a frustum matrix

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



glFrustum Matrix

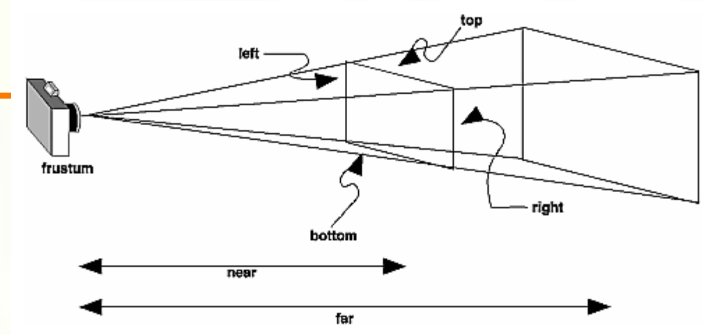


- Projection specification
  - glLoadIdentity();
    glFrustum(-4, +4, -3, +3, 5, 80)
    - left=-4, right=4, bottom=-3, top=3, near=5, far=80
- Matrix

symmetric left/right & top/bottom so zero

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & 0 & 0 & 0 \\ 0 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & -\frac{85}{75} & -\frac{800}{75} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

glFrustum Example



-Z axis

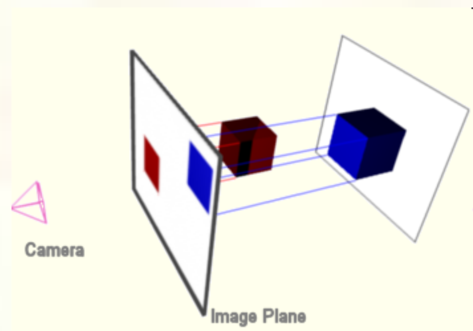
- Consider
  - glLoadIdentity();
    glFrustum(-30, 30, -20, 20, 1, 1000)
    - left=-30, right=30, bottom=-20, top=20, near=1, far=1000
- Matrix

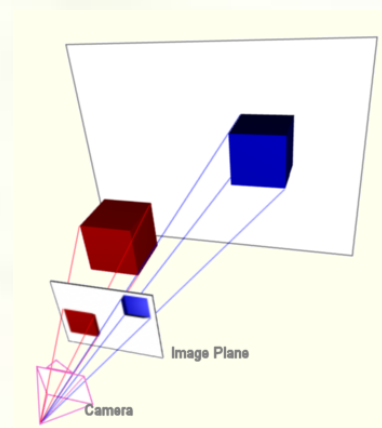
symmetric left/right & top/bottom so zero

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{30} & 0 & 0 & 0 \\ 0 & \frac{1}{20} & 0 & 0 \\ 0 & 0 & -\frac{1001}{999} & -\frac{2000}{999} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

glOrtho and glFrustum

- These OpenGL commands provide a parameterized transform mapping eye space into the "clip cube"
- Each command
  - glOrtho is orthographic
  - glFrustum is single-point perspective



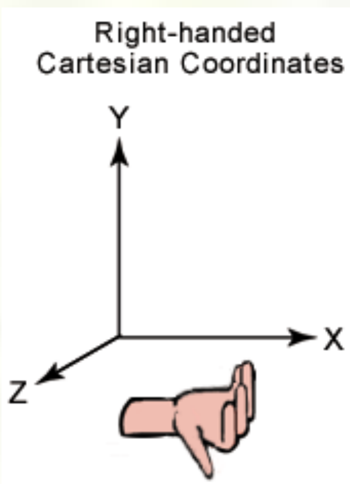


Handedness of Coordinate Systems

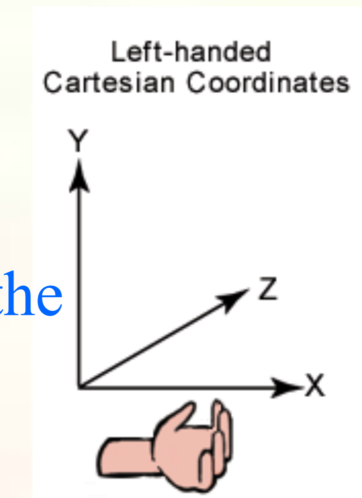
- When
  - Object coordinate system is right-handed,
  - Modelview transform is generated from one or more of the commands glTranslate, glRotate, and glScale with positive scaling values,
  - Projection transform is loaded with glLoadIdentity followed by exactly one of glOrtho or glFrustum,
  - Near value specified for glDepthRange is less than the far value;
- Then
  - Eye coordinate system is right-handed
  - Clip, NDC, and window coordinate systems are left-handed

Conventional OpenGL Handedness

- Right-handed
  - Object space
  - Eye space
- Left-handed
  - Clip space
  - Normalized Device Coordinate (NDC) space
  - Window space



In eye space, eye is "looking down" the negative Z axis



Positive depth is further from vie

Affine Frustum Clip Equations

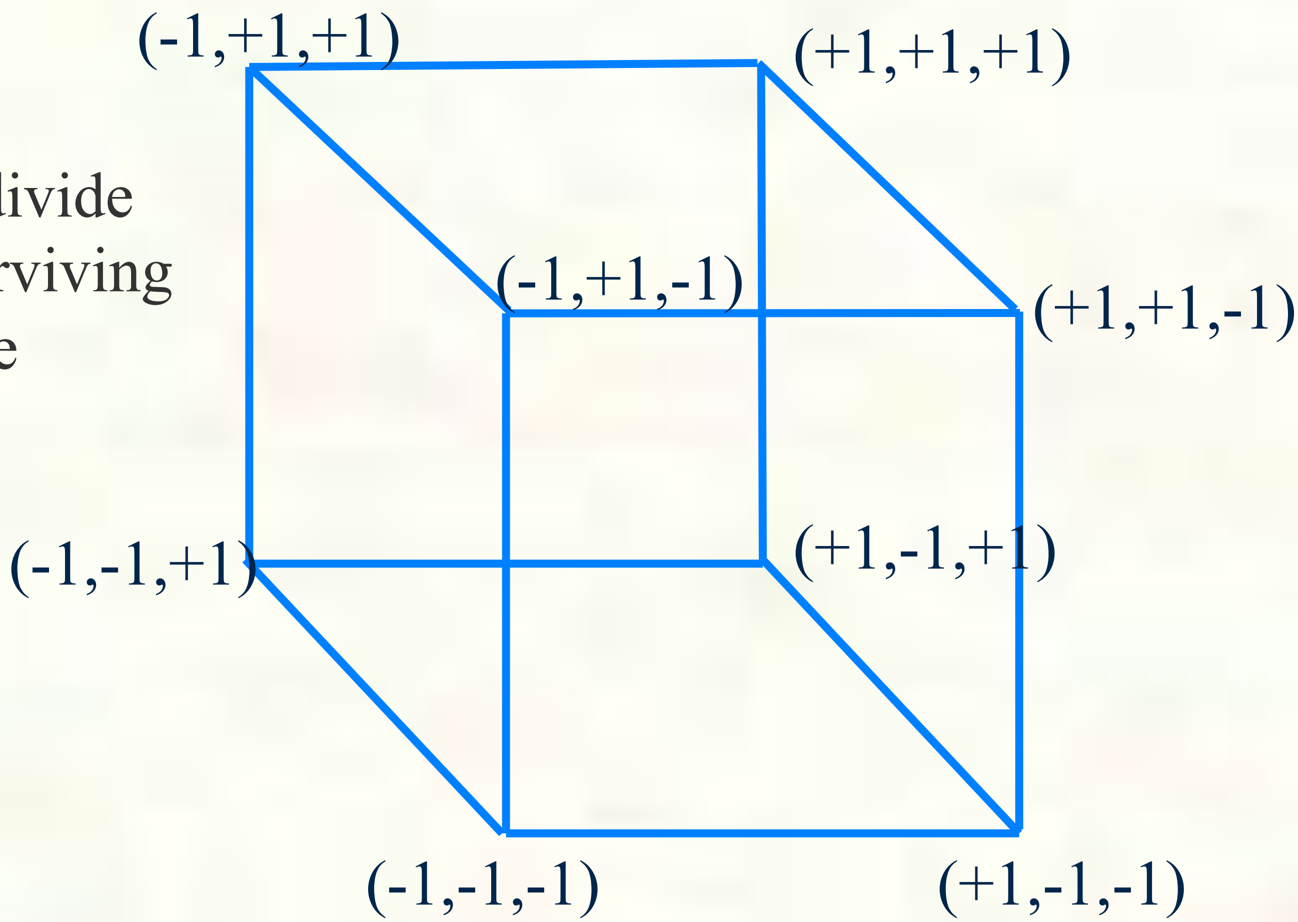
- The idea of a $[-1,+1]^3$ view frustum cube
  - Regions outside this cube get clipped
  - Regions inside the cube get rasterized
- Equations
  - $-1 \leq x_c \leq +1$
  - $-1 \leq y_c \leq +1$
  - $-1 \leq z_c \leq +1$

Projective Frustum Clip Equations

- Generalizes clip cube as a projective space
  - Uses (x_c, y_c, z_c, w_c) clip-space coordinates
- Equations
  - $-w_c \le x_c \le +w_c$
  - $-w_c \le y_c \le +w_c$
  - $-w_c \le z_c \le +w_c$
- Notice
  - Impossible for $w_c < 0$ to survive clipping
  - Interpretation: w_c is distance in front of the eye
    - So negative w_c values are "behind your head"

NDC Space Clip Cube

Post-perspective divide puts the region surviving clipping within the $[-1,+1]^3$



Clip Space Clip Cube

Constraints

$x_{min} = -w$

$x_{max} = w$

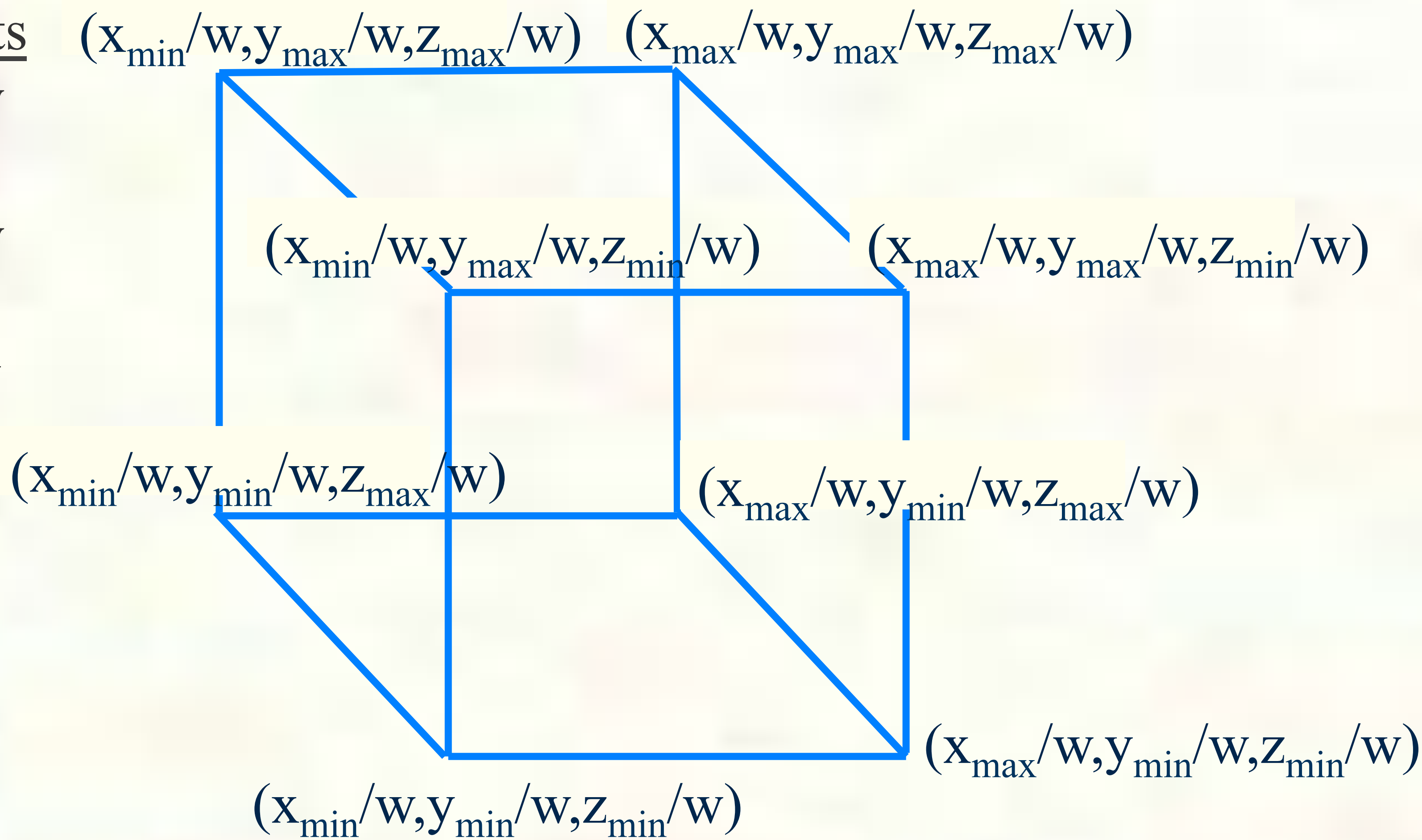
$y_{min} = -w$

$y_{max} = w$

$z_{min} = -w$

$z_{max} = w$

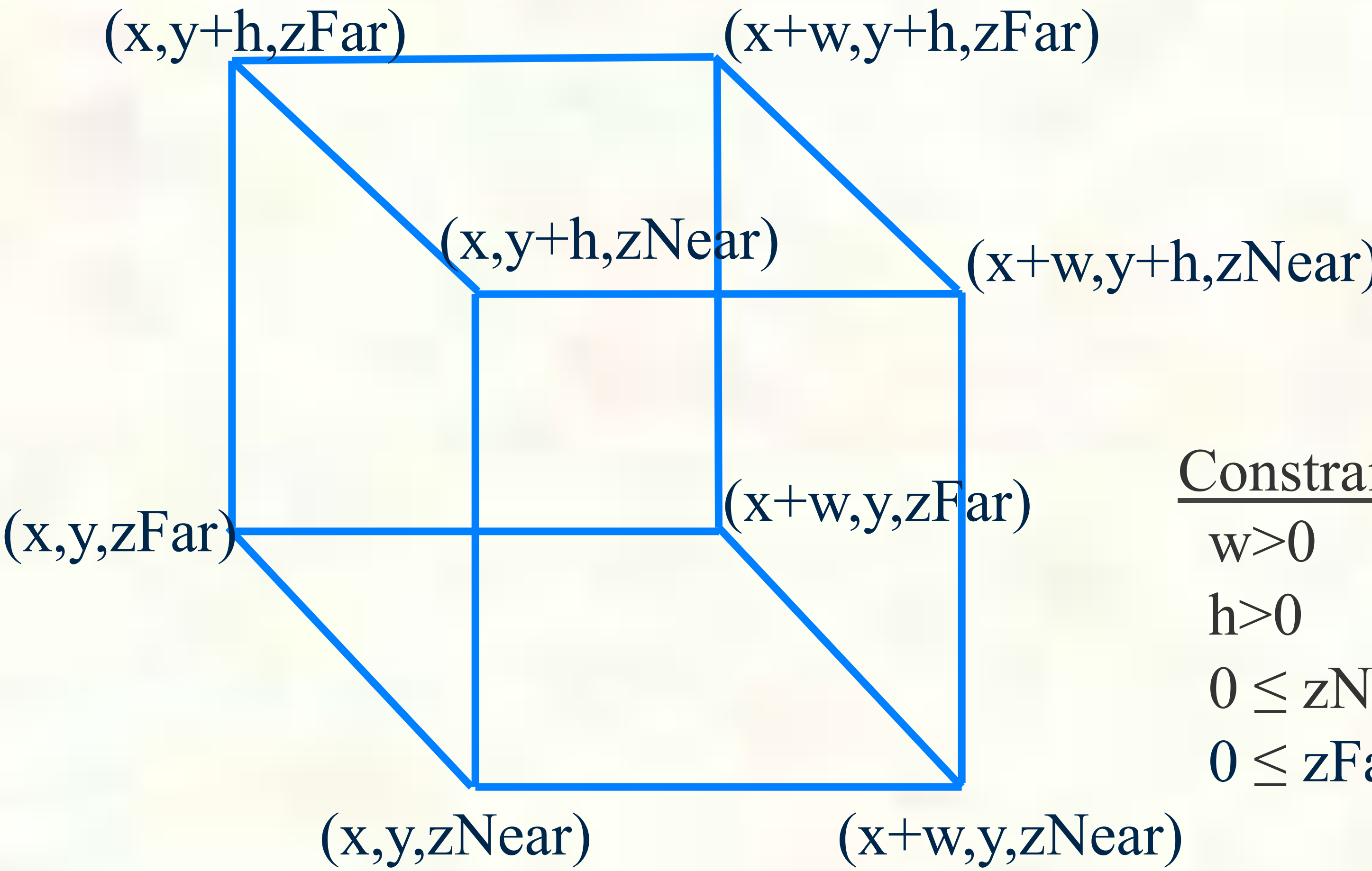
$w>0$



Pre-perspective divide puts the region surviving clipping within

$-w \le x \le w, \quad -w \le y \le w, \quad -w \le z \le w$

Window Space Clip Cube



Constraints

- w>0
- h>0
- $0 \leq zNear \leq 1$
- $0 \leq zFar \leq 1$

Assuming glViewport(x,y,w,h) and glDepthRange(zNear,zFar)

Perspective Divide

- Divide clip-space (x,y,z) by clip-space w
  - To get Normalized Device Coordinate (NDC) space
- Means reciprocal operation is done once
  - And done after clipping
  - Minimizes division by zero concern

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} x_c / w_c \\ y_c / w_c \\ z_c / w_c \end{bmatrix}$$

Transform All Box Corners

keep in mind: looking down the negative Z axis… so Z box coordinates are negative n (-15) and negative f (+25)

- Consider
  - glLoadIdentity();
    glOrtho(-20, 30, 10, 60, 15, -25);
    - l=-20, r=30, b=10, t=50, n=15, f=-25
  - Eight box corners: (-20,10,-15), (-20,10,25), (-20, 50,-15), (-20, 50,-25), (30,10,-15), (30,10,25), (30,50,-15), (30,50,25)
- Transform each corner by the 4x4 matrix

$$\begin{bmatrix} \frac{1}{25} & 0 & 0 & -\frac{1}{5} \\ 0 & \frac{1}{20} & 0 & -\frac{3}{2} \\ 0 & 0 & \frac{1}{20} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -20 & -20 & -20 & -20 & 30 & 30 & 30 & 30 \\ 10 & 10 & 50 & 50 & 10 & 10 & 50 & 50 \\ -15 & 25 & -15 & 25 & -15 & 25 & -15 & 25 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

8 corners in column vector (position) form

Box Corners in Clip Space

8 "eye space" corners in column vector form

$$\begin{bmatrix} \frac{1}{25} & 0 & 0 & -\frac{1}{5} \\ 0 & \frac{1}{20} & 0 & -\frac{3}{2} \\ 0 & 0 & \frac{1}{20} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -20 & -20 & -20 & -20 & 30 & 30 & 30 & 30 \\ 10 & 10 & 50 & 50 & 10 & 10 & 50 & 50 \\ -15 & 25 & -15 & 25 & -15 & 25 & -15 & 25 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

result is "corners" of clip space (and NDC) clip cube

Transform All Box Corners

***keep in mind**: looking down the negative Z axis… so Z box coordinates are negative n (-1) and negative f (-1000)*

- Consider
  - glLoadIdentity();
    glFrustum(-30, 30, -20, 20, 1, 1000)
    - left=-30, right=30, bottom=-20, top=20, near=1, far=1000
  - Eight box corners: (-30,-20,-1), (-30,-20,-1000), (-30, 20,-1), (-30, 20,-1000), (30,10,-1), (30,10,-1000), (30,50,-1), (30,50,-1000)
- Transform each corner by the 4x4 matrix

$$\begin{bmatrix} \frac{1}{30} & 0 & 0 & 0 \\ 0 & \frac{1}{20} & 0 & 0 \\ 0 & 0 & -\frac{1001}{999} & -\frac{2000}{999} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{array}{c} \begin{bmatrix} -30 & -30000 & -30 & -30000 & 30 & 30000 & 30 & 30000 \\ -20 & -20000 & 20 & 20000 & -20 & -20000 & 20 & 20000 \\ -1 & -1000 & -1 & -1000 & -1 & -1000 & -1 & -1000 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{cccccccc} \text{near} & \text{far} & \text{near} & \text{far} & \text{near} & \text{far} & \text{near} & \text{far} \end{array} \end{array}$$

Box Corners in Clip Space

8 "eye space" corners in column vector form

$$\begin{bmatrix} \frac{1}{30} & 0 & 0 & 0 \\ 0 & \frac{1}{20} & 0 & 0 \\ 0 & 0 & -\frac{1001}{999} & -\frac{2000}{999} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -30 & -30000 & -30 & -30000 & 30 & 30000 & 30 & 30000 \\ -20 & -20000 & 20 & 20000 & -20 & -20000 & 20 & 20000 \\ -1 & -1000 & -1 & -1000 & -1 & -1000 & -1 & -1000 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1000 & -1 & -1000 & +1 & +1000 & +1 & +1000 \\ -1 & -1000 & +1 & +1000 & -1 & -1000 & +1 & +1000 \\ -1 & +1000 & -1 & +1000 & -1 & +1000 & -1 & +1000 \\ +1 & +1000 & +1 & +1000 & +1 & +1000 & +1 & +1000 \end{bmatrix}$$

Box Corners in NDC Space

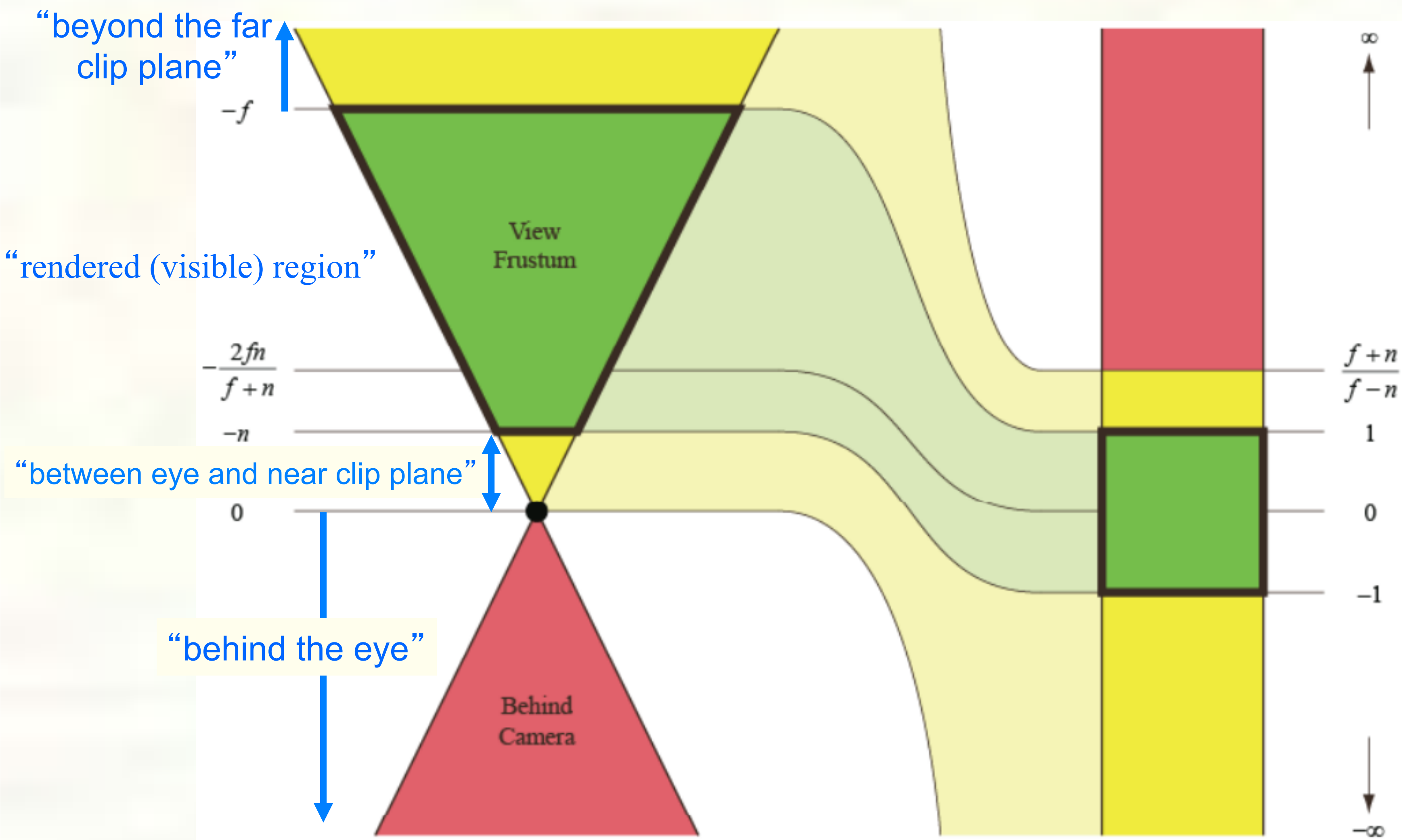
- Perform perspective divide

$$wdivide\left(\begin{bmatrix} -1 & -1000 & -1 & -1000 & +1 & +1000 & +1 & +1000 \\ -1 & -1000 & +1 & +1000 & -1 & -1000 & +1 & +1000 \\ -1 & +1000 & -1 & +1000 & -1 & +1000 & -1 & +1000 \\ +1 & +1000 & +1 & +1000 & +1 & +1000 & +1 & +1000 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\ +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \end{bmatrix}$$

W component is 1 (at near plane) or 1/1000 (at far plane)
Z component is always -1 (assuming W=1 eye-space positions)

Eye Space and NDC Space



[Eric Lengyel]   Eye space   NDC space

Hidden-Surface Removal

- Although our selection of the form of the perspective matrices may appear somewhat arbitrary, it was chosen so that if $z_1 > z_2$ in the original clipping volume then for the transformed points $z_1' > z_2'$
- Thus hidden surface removal works if we first apply the normalization transformation
- However, the formula $z' = -(\alpha+\beta/z)$ implies that the distances are distorted by the normalization which can cause numerical problems especially if the near distance is small

Why do we do it this way?

- Normalization allows for a single pipeline for both perspective and orthogonal viewing
- We stay in four dimensional homogeneous coordinates as long as possible to retain three-dimensional information needed for hidden-surface removal and shading
- We simplify clipping

Notes

- We stay in four-dimensional homogeneous coordinates through both the modelview and projection transformations
  - Both these transformations are nonsingular
  - Default to identity matrices (orthogonal view)
- Normalization lets us clip against simple cube regardless of type of projection
- Delay final projection until end
  - Important for hidden-surface removal to retain depth information as long as possible

Viewport and Depth Range

- Prototypes
  - glViewport(GLint vx, GLint vy, GLsizei vw, GLsizei vh)
  - glDepthRange(GLclampd n, GLclampd f)
- Equations
  - Maps NDC space to window space

$$\begin{pmatrix} x_w \\ y_w \\ z_w \end{pmatrix} = \begin{pmatrix} \frac{v_w}{2} x_n + \left(v_x + \frac{v_w}{2} \right) \\ \frac{v_h}{2} y_n + \left(v_y + \frac{v_h}{2} \right) \\ \frac{f-n}{2} z_n + \frac{f+n}{2} \end{pmatrix}$$

Next Lecture

- Modelview Transformations