# Euler Angles, Quaternions, and Animation

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We have already seen how to represent transformations as $4 \times 4$ homogeneous matrices. In particular, we know how to develop such matrices for rotations about each of the three primary axes and about an arbitrary axis. We will now look more closely at the representation of rotations. In particular, we will be examining *Euler angle* representations and the problem of *gimbal lock*, and we will see how to use an alternative representation in terms of *quaternions* for applications such as animation to avoid problems with gimbal lock.

## 1 Rotations using Euler angles

(Describe yaw, pitch, and roll, gimbal lock, etc. here.)

## 2 Quaternions

Quaternions were invented by William Rowan Hamilton as a three-dimensional generalization of complex numbers. Recall that a complex number $x + i\,y$ can be thought of as a two element vector $(x, y)$ in cartesian space. The first element is called the real part, the second the imaginary part. Suppose we consider a second complex number $a + i\,b$ that is unit length, i.e. for which $a^2 + b^2 = 1$. If we multiply these two complex numbers we get $(a\,x - b\,y) + i(b\,x + a\,y)$. Since $a^2 + b^2 = 1$, we can rename $a = \cos(\theta)$ and $b = \sin(\theta)$ to get $(x\cos(\theta) - y\sin(\theta)) + i(x\sin(\theta) + y\cos(\theta))$ or in vector form $(x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$, i.e. a counterclockwise rotation of the original vector by $\theta$.

A *quaternion* $q = r + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (r, \boldsymbol{v})$ where $\boldsymbol{v} = (x, y, z)$ is the imaginary part, $r$ is the real part, and $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are unit quaternions that are analogous to unit vectors except that they obey the following multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

Note that this implies in addition that:

$$\boldsymbol{ij} = \boldsymbol{k}, \boldsymbol{ji} = -\boldsymbol{k}, \boldsymbol{jk} = \boldsymbol{i}, \boldsymbol{kj} = -\boldsymbol{i}, \boldsymbol{ki} = \boldsymbol{j}, \boldsymbol{ik} = -\boldsymbol{j}$$

From these properties we can derive addition and multiplication rules for quaternions, each of which produces a quaternion.

Addition:

$$q + q' = (r + r', \boldsymbol{v} + \boldsymbol{v}')$$

Multiplication:

$$q\,q' = (r\,r' - \boldsymbol{v} \cdot \boldsymbol{v}', \boldsymbol{v} \times \boldsymbol{v}' + r\boldsymbol{v}' + r'\boldsymbol{v})$$

The dot and cross products used here are the same as those for ordinary vectors. In fact, the definition of quaternion multiplication was the origin of both of these operations.

Recall that every complex number $c = (x, y)$ has a *conjugate* $\bar{c} = (x, -y)$ such that $c\bar{c} = (x^2 + y^2, 0) = |c|^2$. Correspondingly the conjugate of a quaternion $q$ is $\bar{q} = (r, -\boldsymbol{v})$ such that $q\bar{q} = (r^2 + \boldsymbol{v} \cdot \boldsymbol{v}, 0) = |q|^2$. If $|q| = 1$ then $q$ is called a *unit quaternion*, and just as for complex numbers, it is these unit quaternions that we wish to use to represent rotations in three space.

Let's first note that any quaternion has a multiplicative inverse $q^{-1}$ such that $q q^{-1} = 1$. Since $|q|^2 = q\bar{q}$, $\frac{q\bar{q}}{|q|^2} = 1 = q q^{-1}$ and thus $q^{-1} = \frac{\bar{q}}{|q|^2}$. For unit quaternions, $|q|^2 = 1$ so we have $q^{-1} = \bar{q}$.

# 3 Rotation about an arbitrary axis

In order to see how to represent rotations by quaternions, let's first develop a formula for rotation about an arbitrary axis in a form that will make this easy. Suppose we want to rotate a vector $\boldsymbol{v}$ about an arbitrary axis represented by vector $\boldsymbol{\alpha}$. We will begin by decomposing $\boldsymbol{v}$ into perpendicular components $\boldsymbol{v}_{\parallel}$ parallel to $\boldsymbol{\alpha}$ and $\boldsymbol{v}_{\perp}$ perpendicular to $\boldsymbol{\alpha}$. We proceed by rotating $\boldsymbol{v}_{\perp}$ by the desired angle $\theta$ in the plane orthogonal to $\boldsymbol{\alpha}$. Recall that a two dimensional rotation of this sort is given by

$$\boldsymbol{v}'_{\perp} = \begin{bmatrix} s' \\ t' \end{bmatrix} = \begin{bmatrix} s\cos(\theta) - t\sin(\theta) \\ t\cos(\theta) + s\sin(\theta) \end{bmatrix} = \begin{bmatrix} s & -t \\ t & s \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = [\boldsymbol{v}_{\perp}\ \boldsymbol{v}^p_{\perp}] \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \boldsymbol{v}_{\perp}\cos(\theta) + \boldsymbol{v}^p_{\perp}\sin(\theta)$$

in terms of local plane coordinates $(s, t)$. The same formula holds when the vectors are expressed in three dimensional coordinates, these are obtained by $\boldsymbol{v}_{\parallel} = (\boldsymbol{v}\cdot\boldsymbol{\alpha})\boldsymbol{\alpha}$, $\boldsymbol{v}_{\perp} = \boldsymbol{v} - \boldsymbol{v}_{\parallel} = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\alpha})\boldsymbol{\alpha}$, and $\boldsymbol{v}^p_{\perp} = \boldsymbol{\alpha}\times\boldsymbol{v}_{\perp} = \boldsymbol{\alpha}\times\boldsymbol{v}$. Now the complete rotation of $v$ in three space is

$$\begin{aligned} \boldsymbol{v}' &= \boldsymbol{v}_{\parallel} + \boldsymbol{v}'_{\perp} = (\boldsymbol{v}\cdot\boldsymbol{\alpha})\boldsymbol{\alpha} + (\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\alpha})\boldsymbol{\alpha})\cos(\theta) + (\boldsymbol{\alpha}\times\boldsymbol{v})\sin(\theta) \\ &= \boldsymbol{v}\cos(\theta) + ((\boldsymbol{v}\cdot\boldsymbol{\alpha})\boldsymbol{\alpha})(1 - \cos(\theta)) + (\boldsymbol{\alpha}\times\boldsymbol{v})\sin(\theta) \end{aligned}$$

This construction is shown in Figure 1.

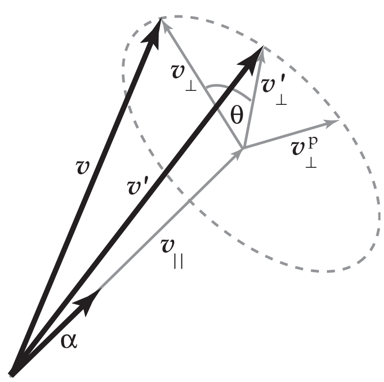


**Figure 1.** Rotation about an arbitrary axis

# 4 Rotation with quaternions

Now we will perform the same rotation using quaternions. Just as with complex numbers, we will let a unit quaternion represent the desired rotation. A unit quaternion $q = r + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ has norm $q\bar{q} = r^2 + x^2 + y^2 + z^2 = r^2 + \boldsymbol{v}\cdot\boldsymbol{v} = 1$. Now $\boldsymbol{v}$, like any other vector, can be represented as $s\,\boldsymbol{n}$ for some scalar $s$ and unit vector $\boldsymbol{n}$ with the same orientation as $\boldsymbol{v}$. For a unit quaternion, then, $r^2 + s^2 = r^2 + s^2\,\boldsymbol{n}\cdot\boldsymbol{n} = r^2 + \boldsymbol{v}\cdot\boldsymbol{v} = 1$. Thus $r = \cos(\phi)$ and $s = \sin(\phi)$ for some $\phi$, and a unit quaternion $q = (\cos(\phi), \sin(\phi)\boldsymbol{n})$ describes an angle $\phi$ and an axis $\boldsymbol{n}$. This suggests that we may be able to use a unit quaternion to rotate a point or vector represented as another quaternion by an angle $\phi$ about axis $\boldsymbol{n}$ by multiplying the two quaternions, just as we could rotate a point or vector represented by a complex number by multiplying it by a unit complex number. It turns out that something of this sort does indeed occur, but it isn't quite this simple.

How convenient that quaternions, like homogeneous points and vectors, have four elements. We can represent a homogeneous three dimensional point or vector as a quaternion by letting the homogeneous coordinate be the real part of the quaternion and the $x$, $y$ and $z$ coordinates be the components of the vector part of the quaternion. Thus homogeneous point $p_h = (x, y, z, 1)$ is represented as $p_q = (1, x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k})$ and homogeneous vector $\boldsymbol{v}_h = (x, y, z, 0)$ becomes $\boldsymbol{v}_q = (0, x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}) = (0, \boldsymbol{v})$. If we want to rotate this point or vector about an axis $\boldsymbol{\alpha}$, we will use the quaternion $r = (\cos(\phi), \sin(\phi)\boldsymbol{\alpha})$. The rotation is performed by the operation $\boldsymbol{v}'_q = r\,\boldsymbol{v}_q\,\bar{r}$ in quaternion space.

Let's expand this operation to verify that a rotation is performed and to determine what rotation it is. We'll first derive a general expression for $t' = s\,t\,\bar{s}$ for arbitrary quaternion $t = (b, \boldsymbol{v})$ and unit quaterion $s = (c, s\,\boldsymbol{n})$ for unit vector $\boldsymbol{n}$.

$$
\begin{aligned}
t' = s\,t\,\bar{s} &= (c, s\,\boldsymbol{n})(b, \boldsymbol{v})(c, -s\,\boldsymbol{n}) \\
&= (b\,c - s\,\boldsymbol{n}\cdot\boldsymbol{v}, s\,\boldsymbol{n}\times\boldsymbol{v} + c\,\boldsymbol{v} + b\,s\,\boldsymbol{n})(c, -s\,\boldsymbol{n}) \\
&= (b\,c^2 - c\,s\,\boldsymbol{n}\cdot\boldsymbol{v} + s^2\boldsymbol{n}\cdot(\boldsymbol{n}\times\boldsymbol{v}) + c\,s\,\boldsymbol{n}\cdot\boldsymbol{v} + b\,s^2\,\boldsymbol{n}\cdot\boldsymbol{n}, \\
&\quad s^2\,\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{v}) + c\,s(\boldsymbol{n}\times\boldsymbol{v}) - b\,s^2\,(\boldsymbol{n}\times\boldsymbol{n}) - b\,c\,s\,\boldsymbol{n} + s^2(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} + c\,s\,(\boldsymbol{n}\times\boldsymbol{v}) + c^2\,\boldsymbol{v} + b\,c\,s\,\boldsymbol{n}) \\
&= (b(c^2 + s^2\boldsymbol{n}\cdot\boldsymbol{n}), s^2(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} - s^2(\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{v} + 2c\,s(\boldsymbol{n}\times\boldsymbol{v}) + s^2(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} + c^2\,\boldsymbol{v}) \\
&= (b, 2c\,s(\boldsymbol{n}\times\boldsymbol{v}) + 2s^2(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} + (c^2 - s^2)\boldsymbol{v})
\end{aligned}
$$

Recalling that for a unit quaternion, $c$ is $\cos(\phi)$ and $s$ is $\sin(\phi)$ for some $\phi$, this gives us

$$
\begin{aligned}
t' &= (b, 2\cos(\phi)\sin(\phi)(\boldsymbol{n}\times\boldsymbol{v}) + 2\sin^2(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} + (\cos^2(\phi) - \sin^2(\phi))\boldsymbol{v}) \\
&= (b, \cos(2\phi)\boldsymbol{v} + (1 - \cos(2\phi))(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} + \sin(2\phi)(\boldsymbol{n}\times\boldsymbol{v}))
\end{aligned}
$$

The vector or imaginary part of $t'$, which is the three space portion of a quaternion that represents a homogeneous point or vector, is precisely the formula we obtained above for rotation of a vector counterclockwise through angle $\theta = 2\phi$ about axis $\boldsymbol{\alpha} = \boldsymbol{n}$ from the point of view of an observer looking down $\boldsymbol{\alpha}$ toward the origin. The real or scalar part of $t'$ is just $b$, the real or scalar part of $t$ before rotation. Thus this operation preserves the scalar term, which is the homogeneous coordinate of the point or vector represented. Clearly then, unit quaternions rotate both homogeneous points and vectors through twice the angle they encode.

## 5 Interpolating orientations using quaternions

(Representing orientations of objects, but not camera positions. Slerps.)