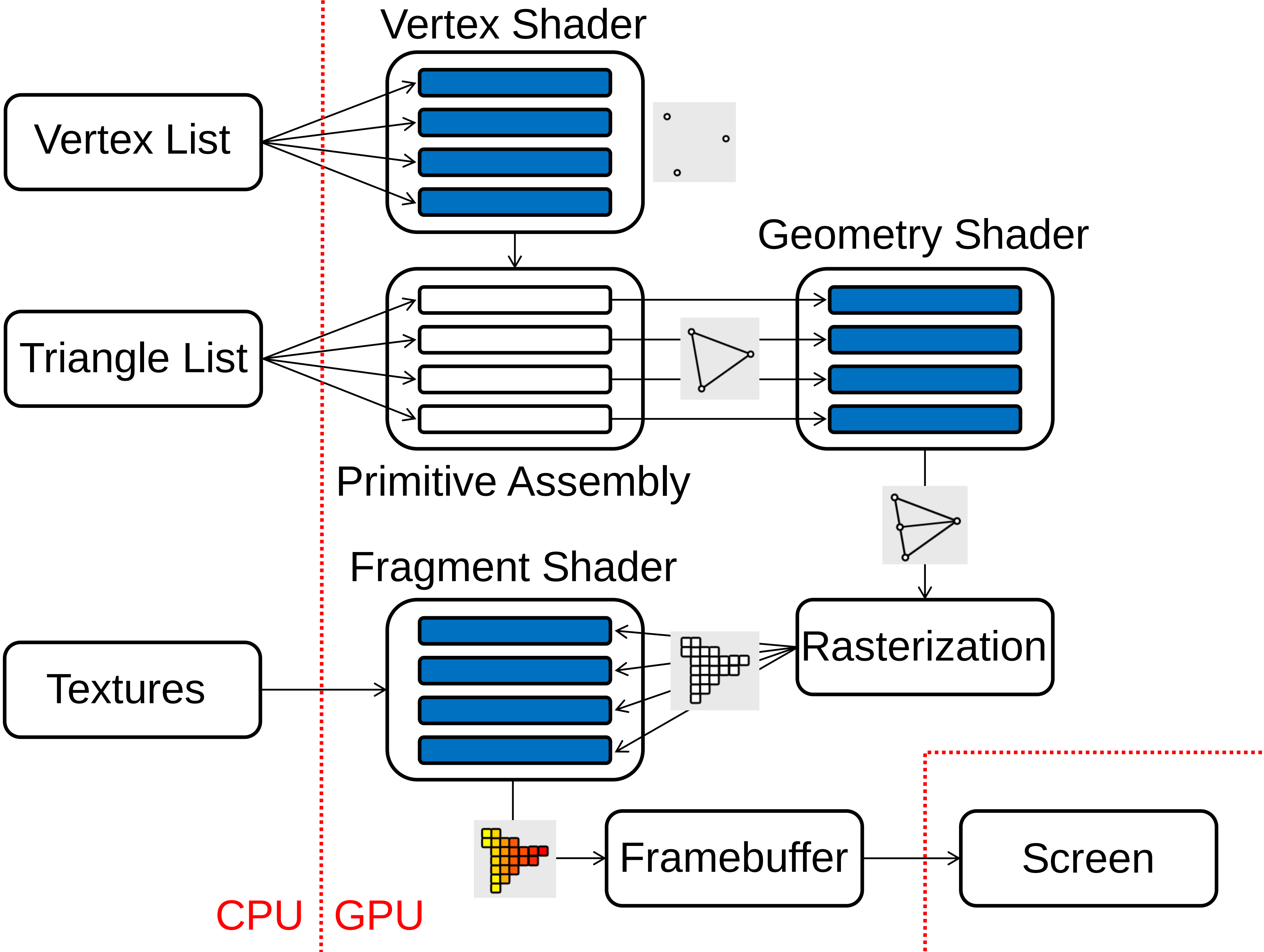
Projections

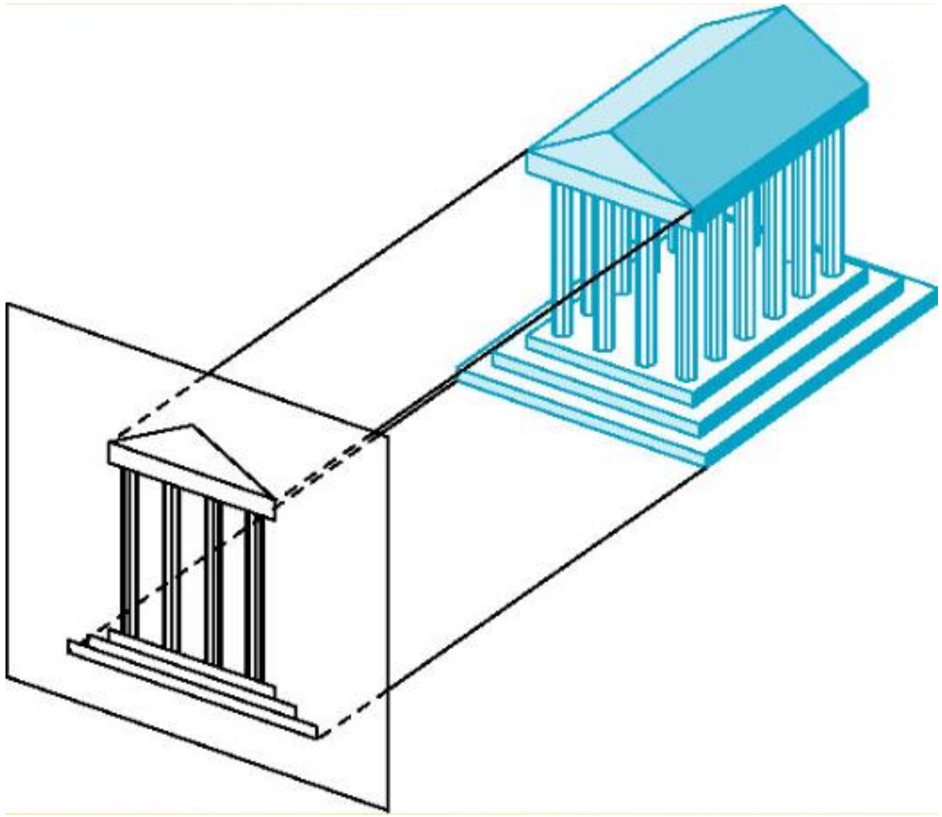


Vertex Shader
Vertex List
Geometry Shader
Triangle List
Primitive Assembly
Fragment Shader
Rasterization
Textures
Framebuffer
Screen
CPU
GPU

Orthographic Projection

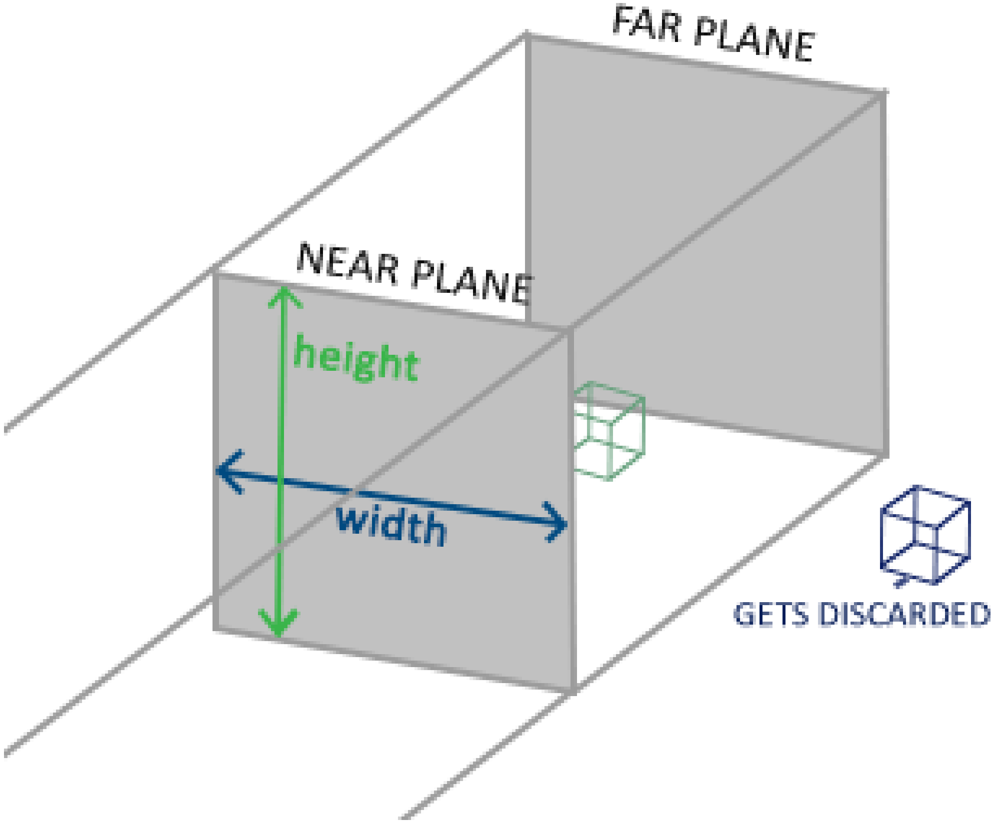
Parallel-project everything onto viewing plane

No perspective



Orthographic Camera Params

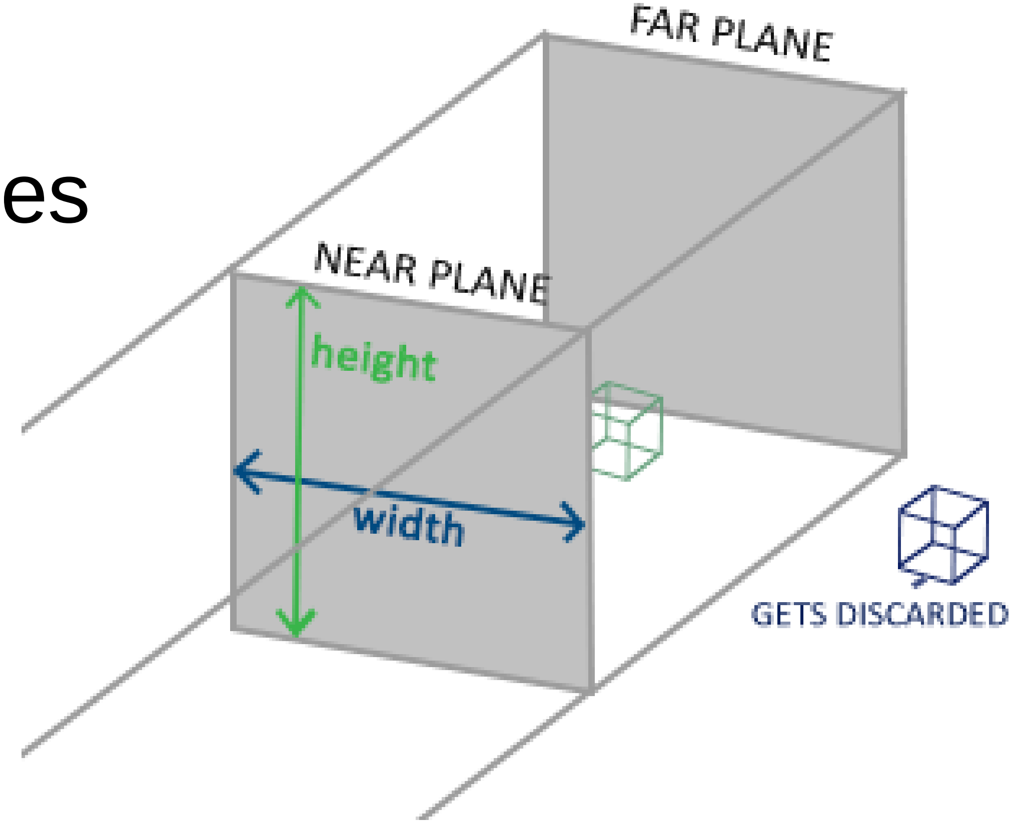
Viewport **width** and **height**



Orthographic Camera Params

Viewport **width** and **height**

Near and **far** planes

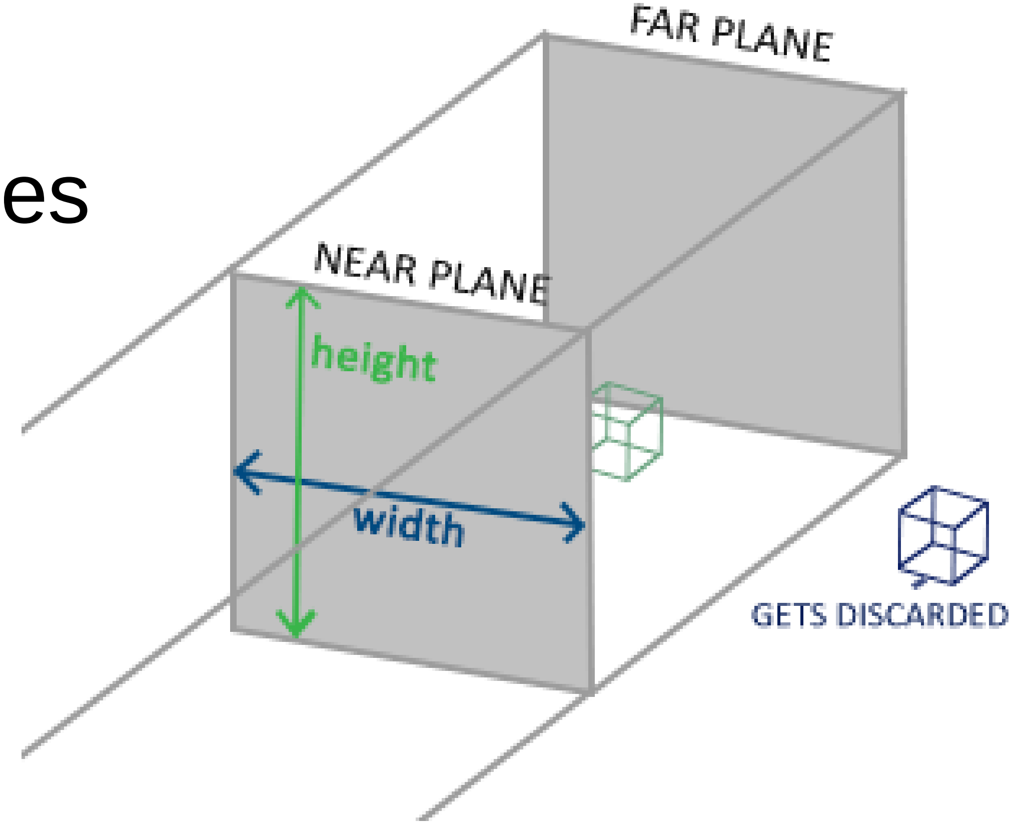


Orthographic Camera Params

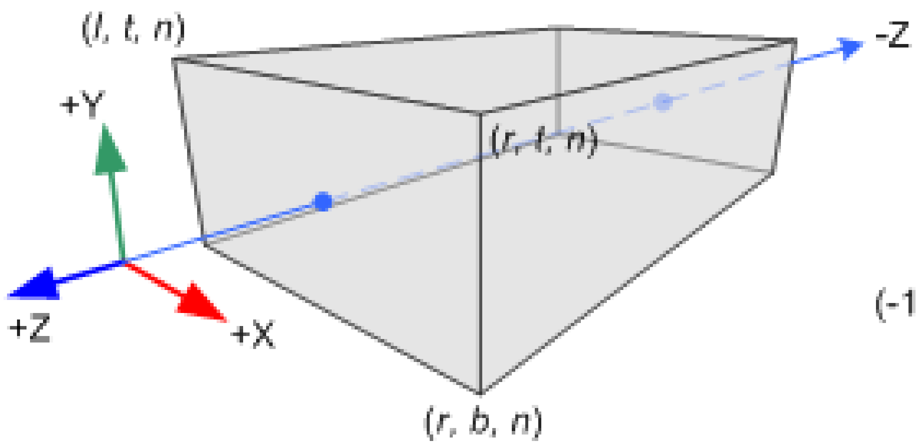
Viewport **width** and **height**

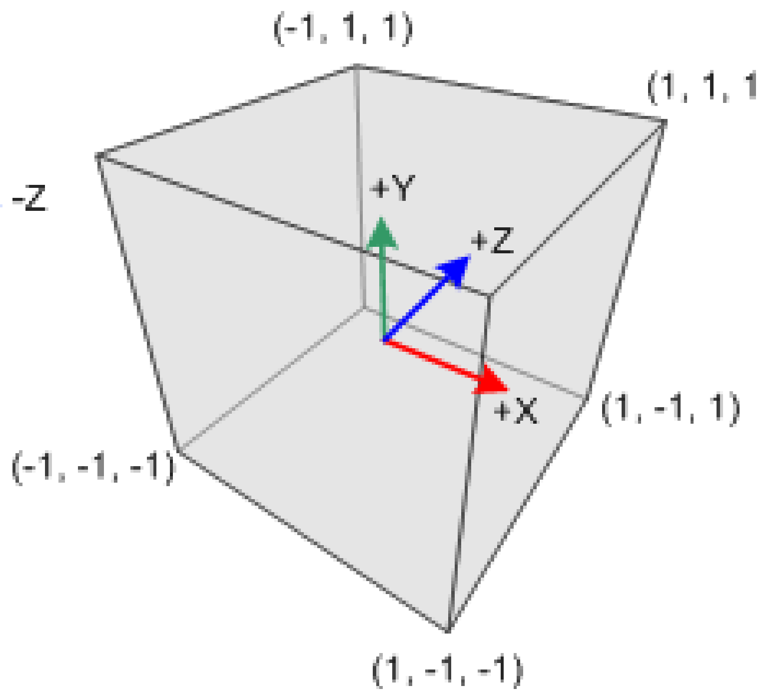
Near and **far** planes

- (why needed)?



Orthographic Projection





Valid X range: $[l, r]$

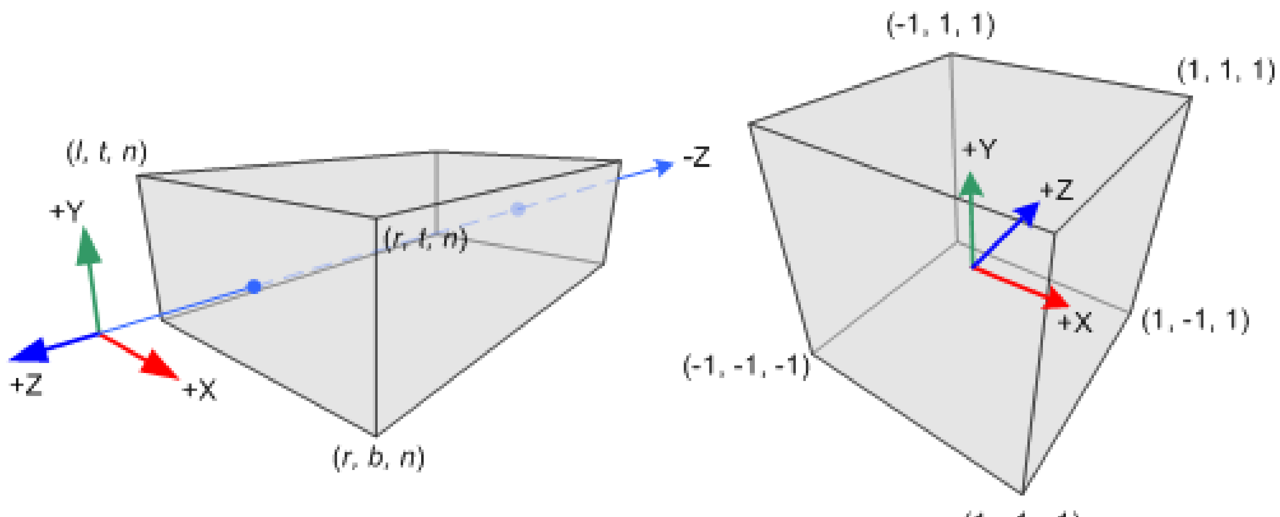
Valid Y range: $[b, t]$

Valid Z range: $[-f, -n]$

Orthographic Projection

Transforms needed:

- translate axes
- flip Z axis
- scale axes

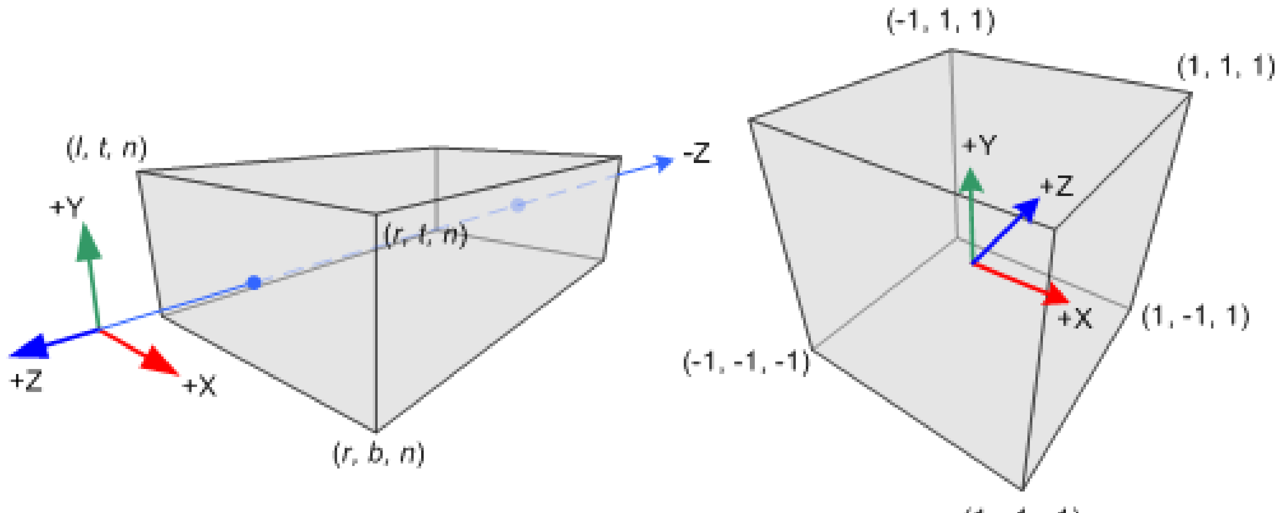


Orthographic Projection

Transforms needed:

- **translate axes**
- flip Z axis
- scale axes

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{l+r}{2} \\ 0 & 1 & 0 & -\frac{b+t}{2} \\ 0 & 0 & 1 & \frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

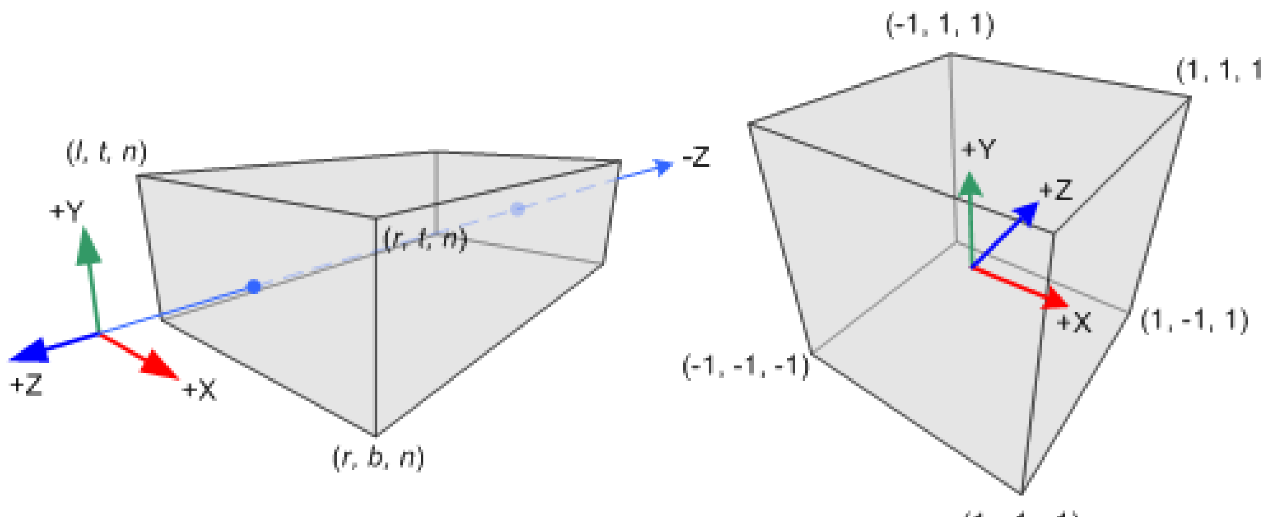


Orthographic Projection

Transforms needed:

- translate axes
- **flip Z axis**
- scale axes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

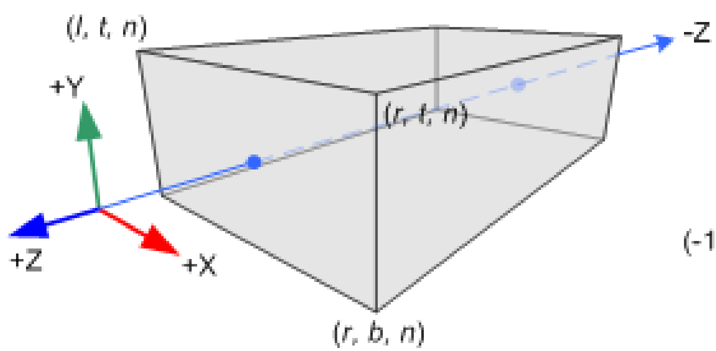


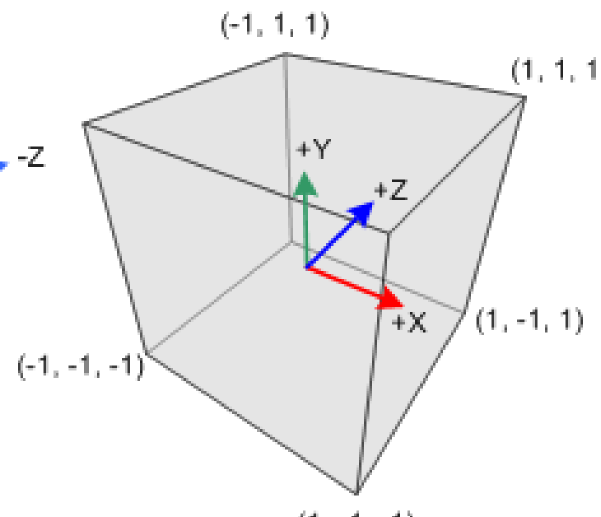
Orthographic Projection

Transforms needed:

- translate axes
- flip Z axis
- **scale axes**

$$\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Orthographic Projection

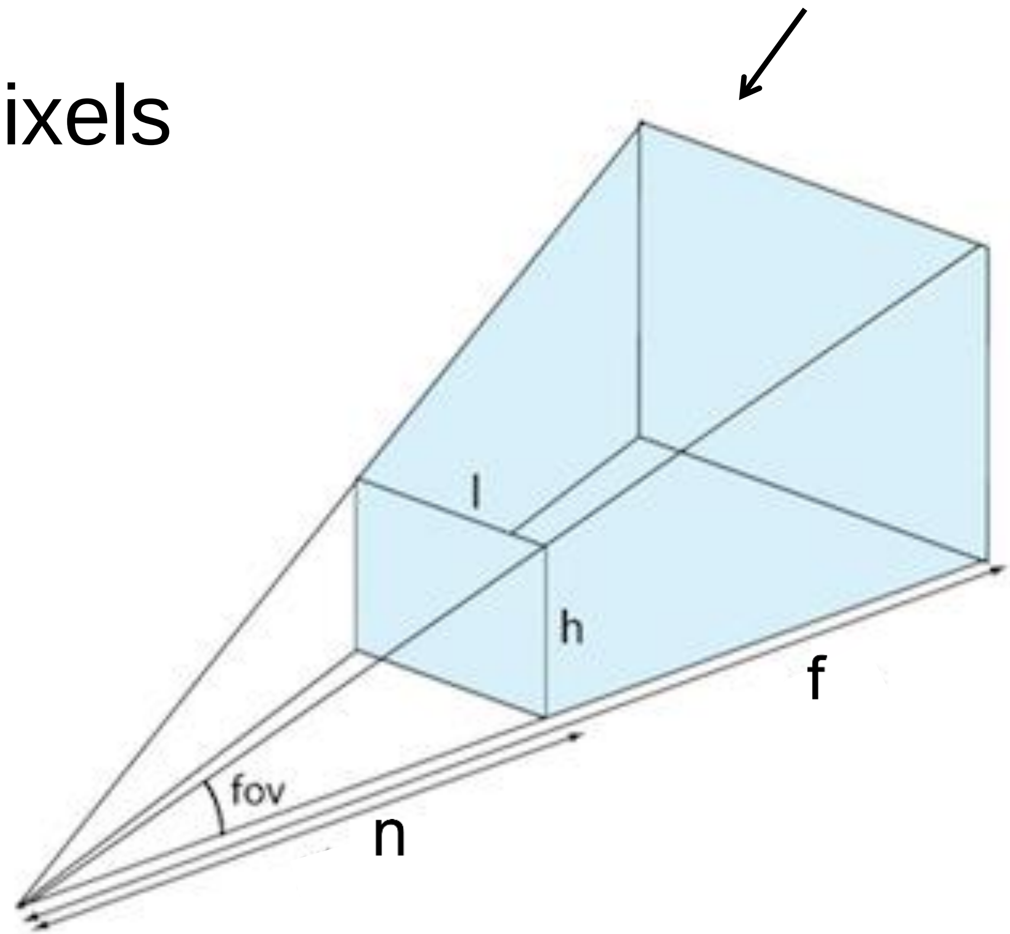
$$
\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & -\frac{l+r}{2} \\ 0 & 1 & 0 & -\frac{b+t}{2} \\ 0 & 0 & 1 & \frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Perspective Camera

“view frustum”

Lines map to pixels

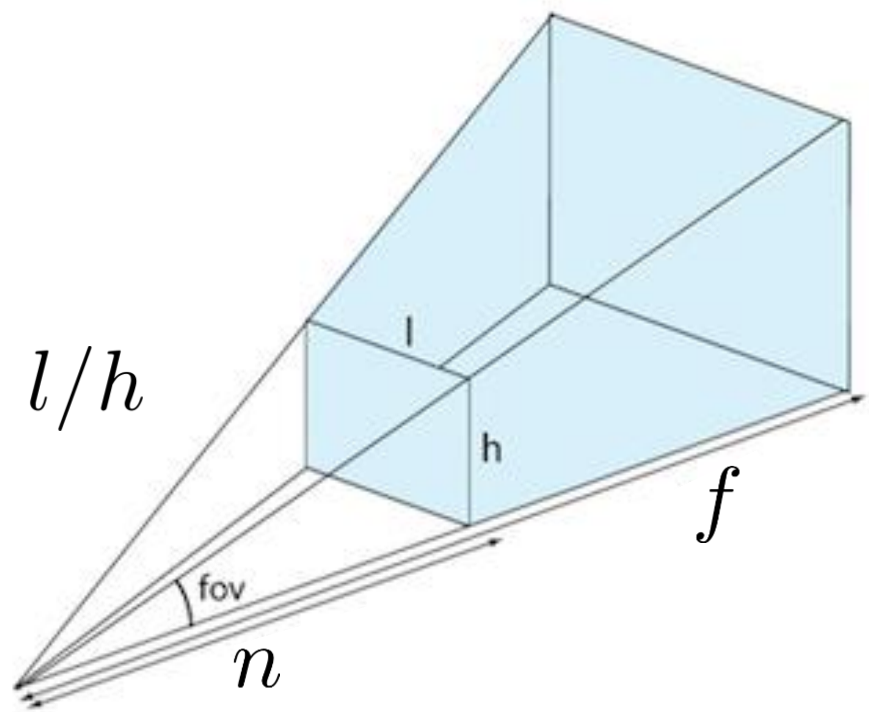


Perspective Camera

Lines map to pixels

Parameters:

- near, far plane
- aspect ratio $a = l/h$
- field of view θ



Field of View

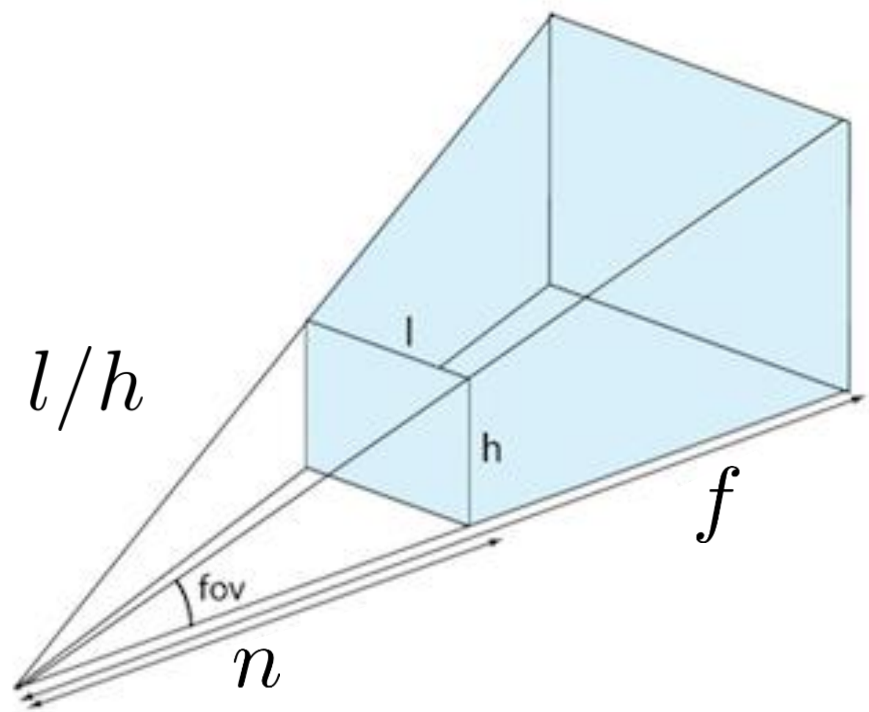
Perspective Camera

Lines map to pixels

$$h = 2n\tan(\theta/2)$$
$$l = 2an\tan(\theta/2)$$

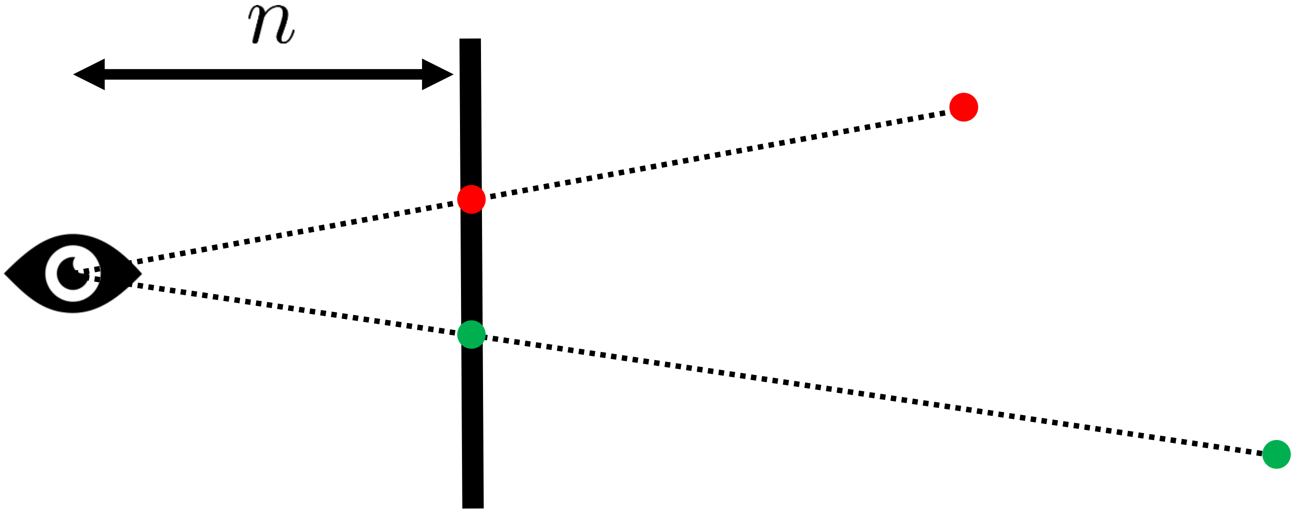
Parameters:

- near, far plane
- aspect ratio $a = l/h$
- field of view θ



Perspective Camera

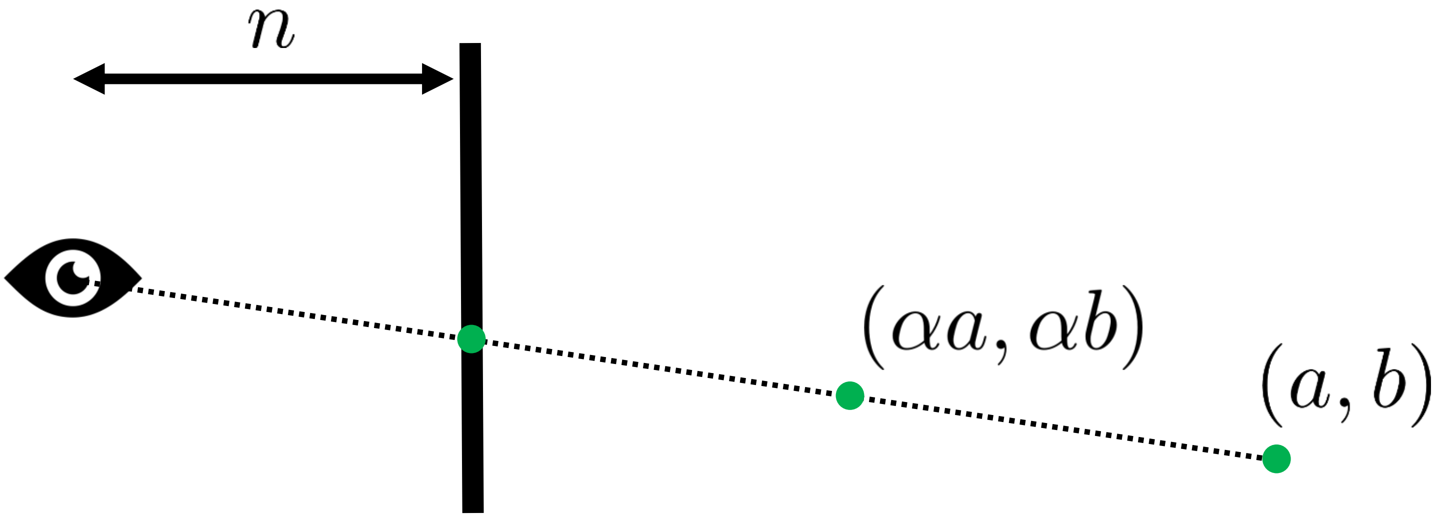
Problem: perspective projection not linear



(Why?)

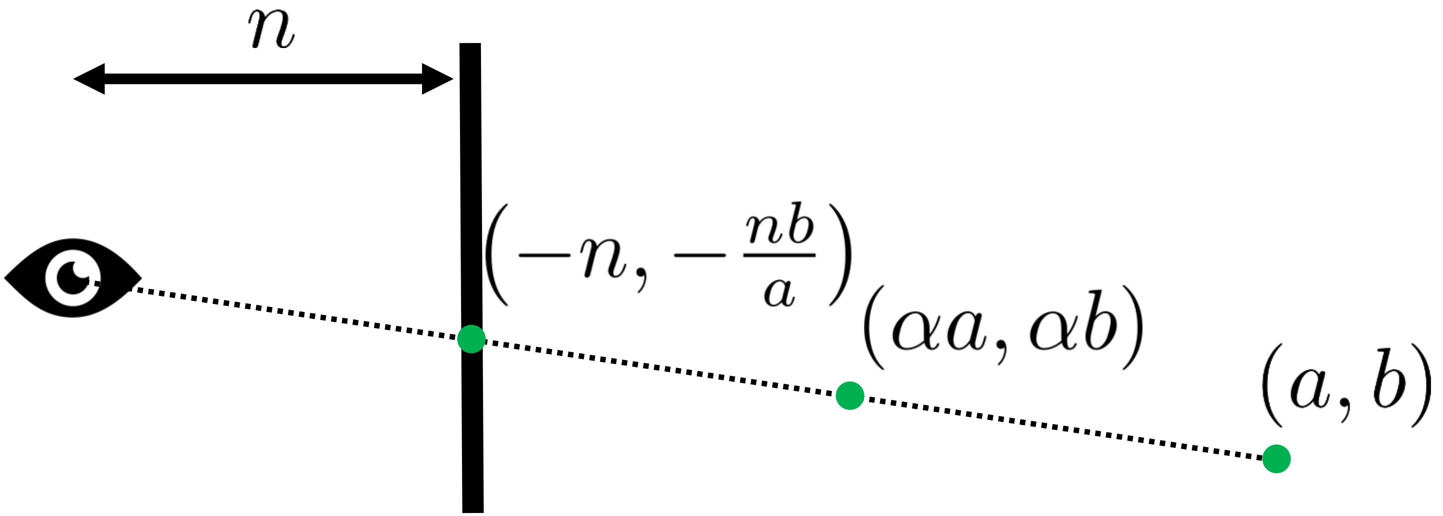
Idea #1: Treat Points as Lines

Using homogeneous coordinate rep:



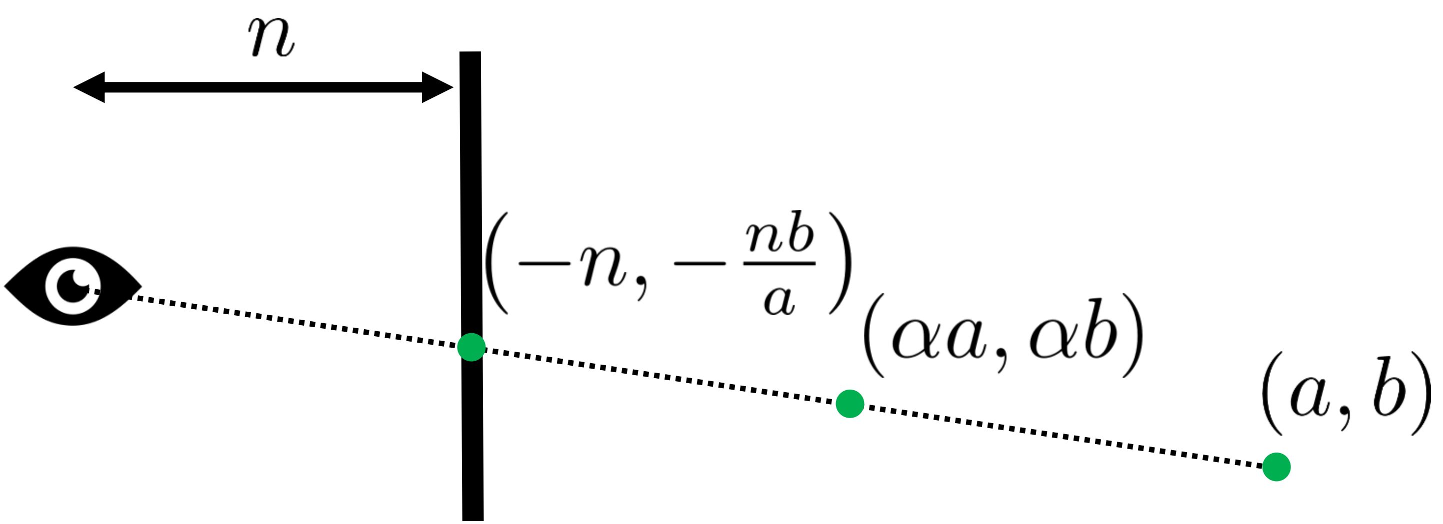
Idea #1: Treat Points as Lines

Using homogeneous coordinate rep:



Idea #1: Treat Points as Lines

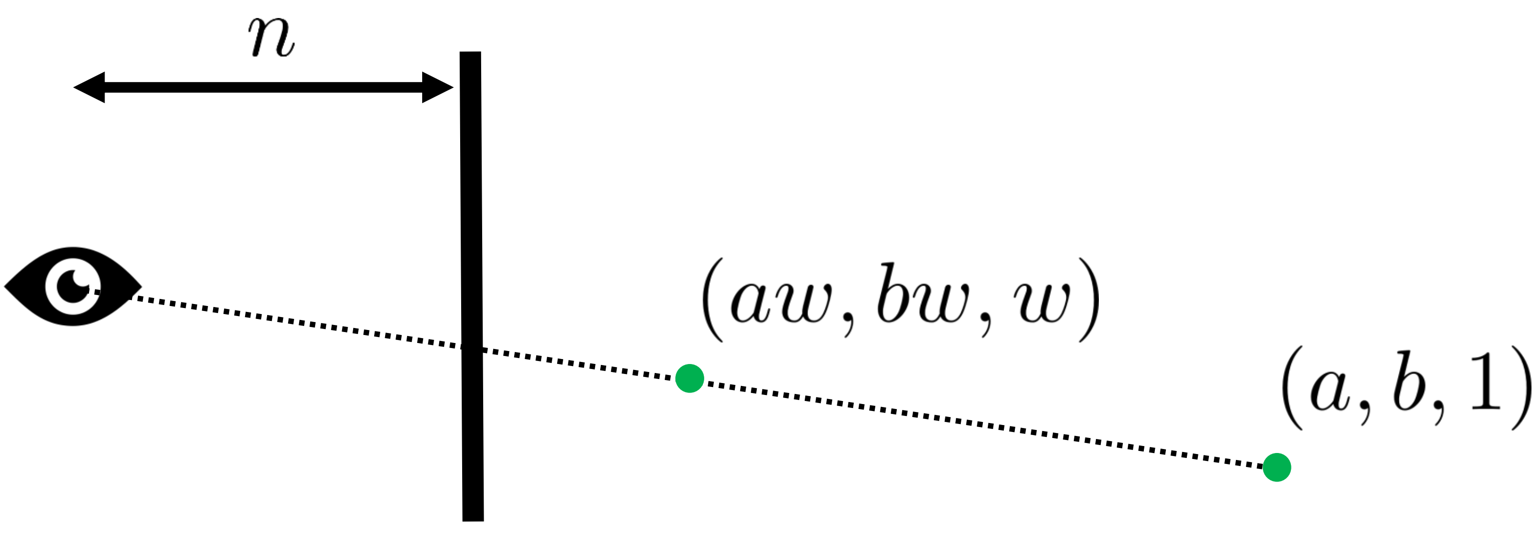
Using homogeneous coordinate rep:



What is the problem?

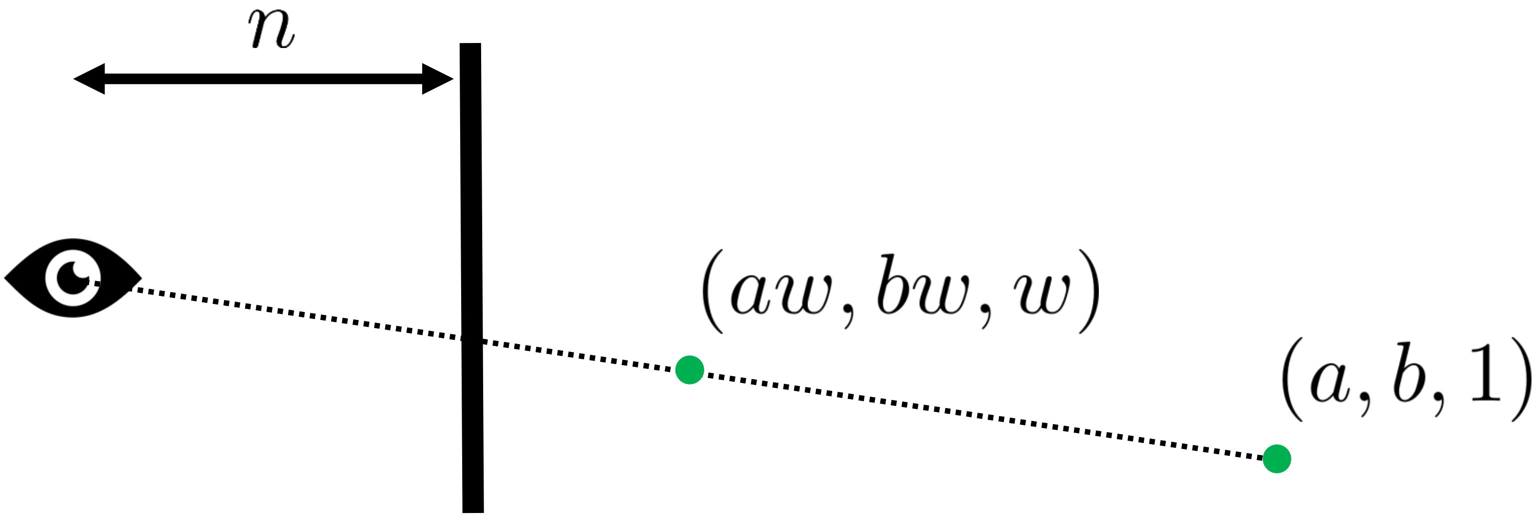
Idea #2: Use One Extra Dimension

Using homogeneous coordinate rep:



Idea #2: Use One Extra Dimension

Using homogeneous coordinate rep:

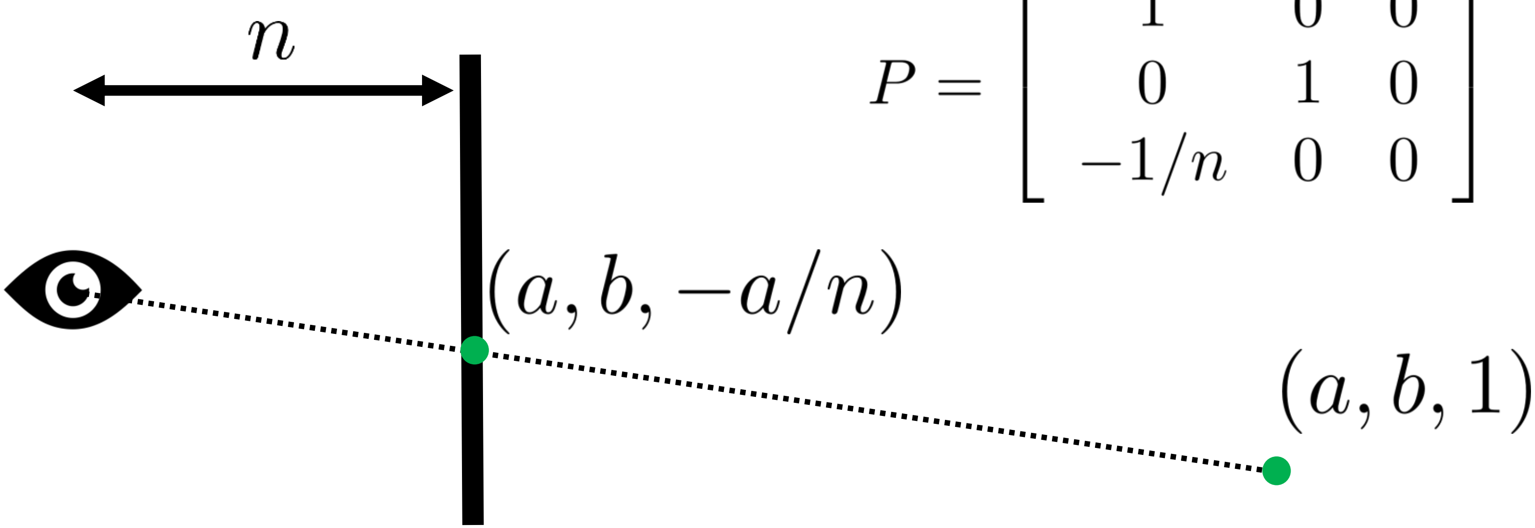


How to do projection onto image plane?

Idea #2: Use One Extra Dimension

Using homogeneous coordinate rep:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/n & 0 & 0 \end{bmatrix}$$

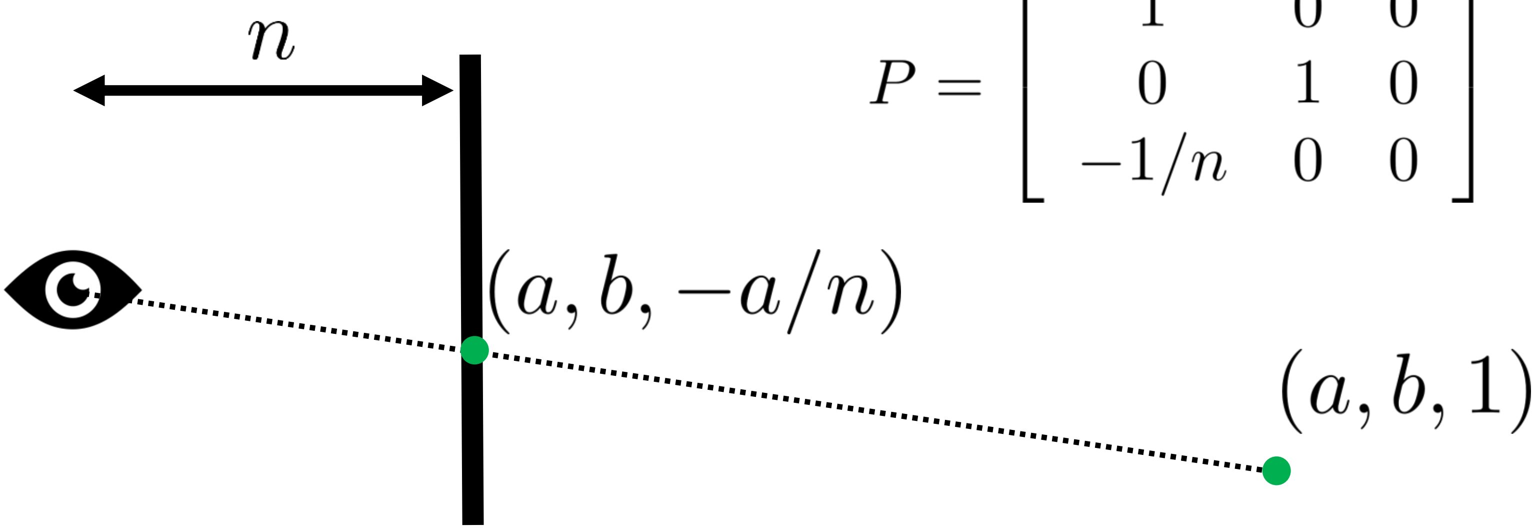


How to do projection onto image plane?

Idea #2: Use One Extra Dimension

Using homogeneous coordinate rep:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/n & 0 & 0 \end{bmatrix}$$

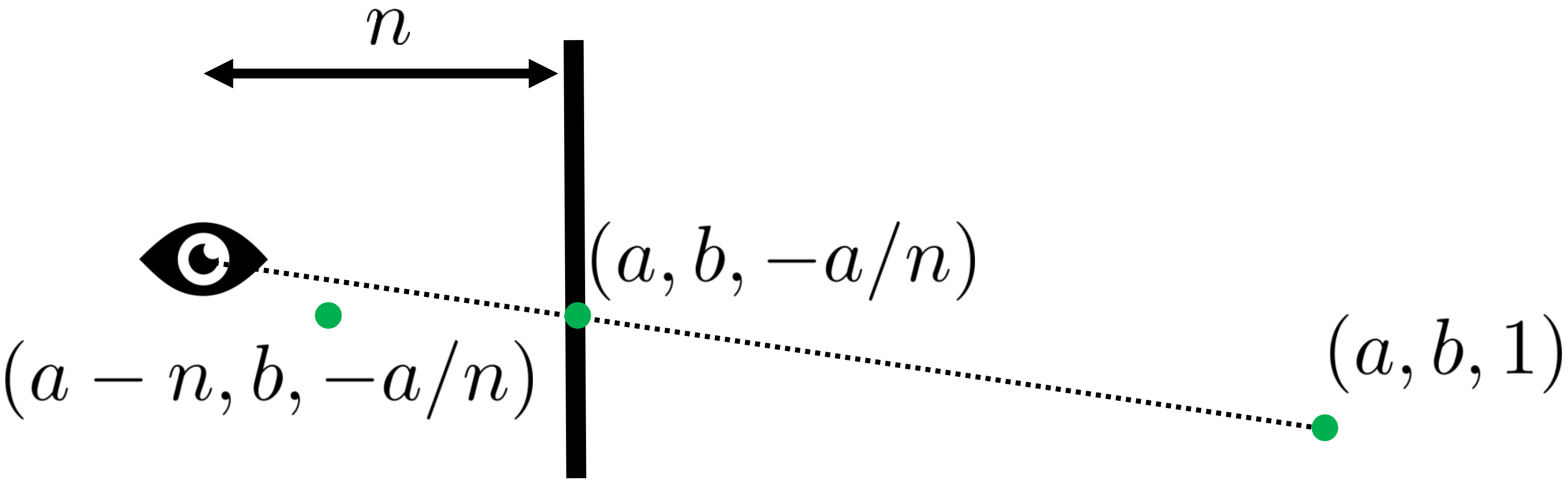


How to do projection onto image plane?

How to preserve depth?

Idea #3: Also Translate

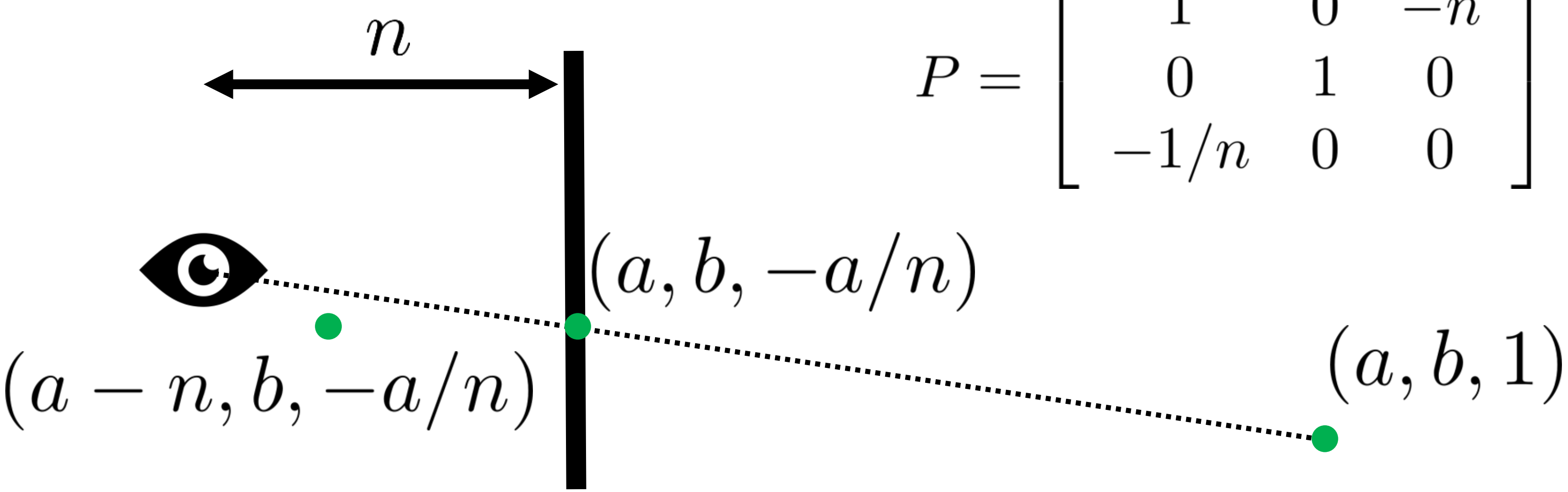
Using homogeneous coordinate rep:



Idea #3: Also Translate

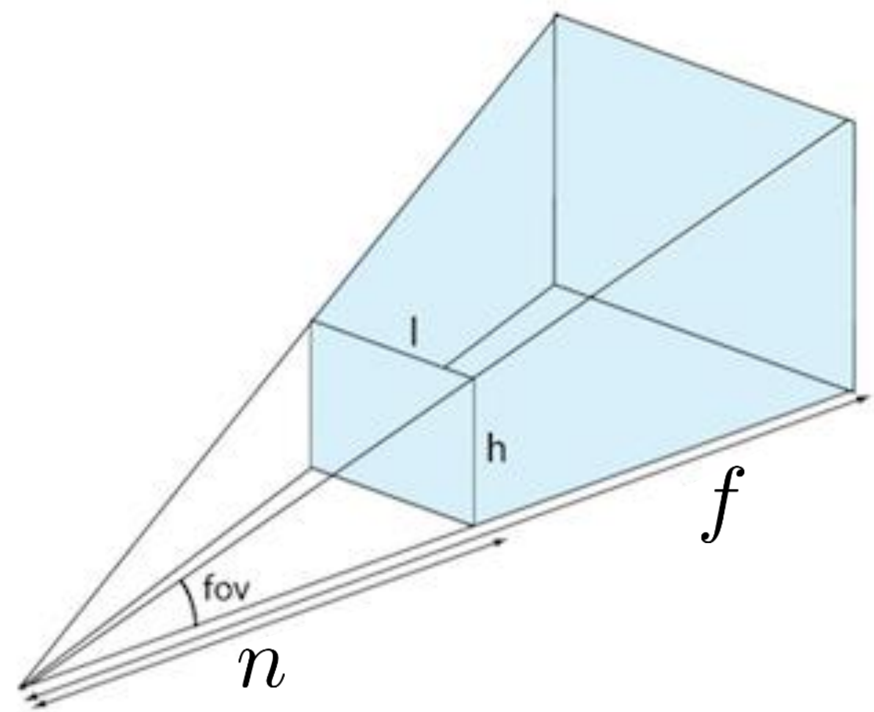
Using homogeneous coordinate rep:

$$P = \begin{bmatrix} 1 & 0 & -n \\ 0 & 1 & 0 \\ -1/n & 0 & 0 \end{bmatrix}$$



Perspective Transformation

$$h = 2n\tan(\theta/2)$$
$$l = 2an\tan(\theta/2)$$



Perspective Transformation

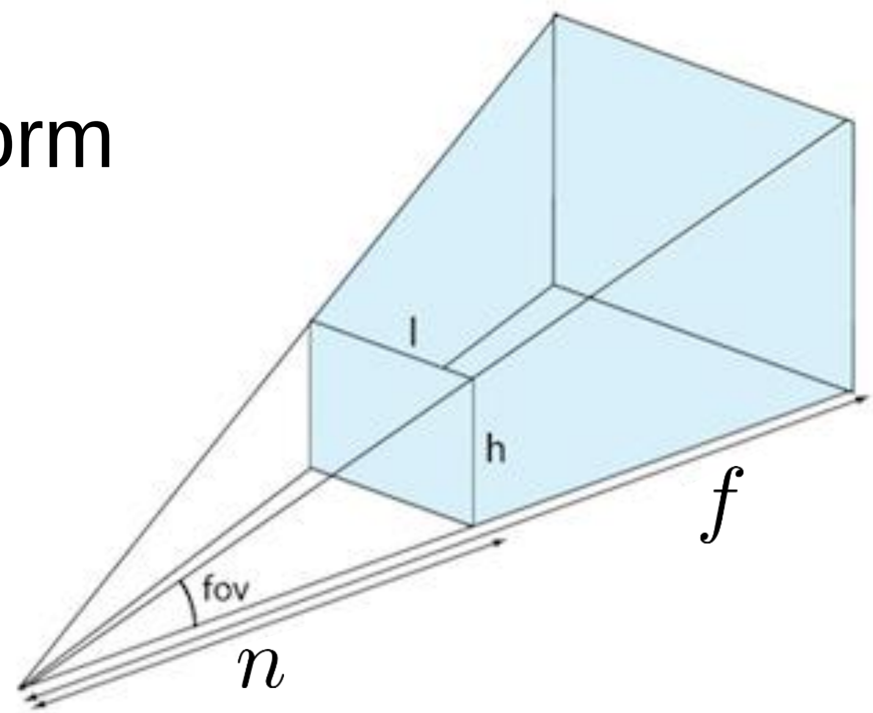
Now in 3D:

$$h = 2n\tan(\theta/2)$$
$$l = 2an\tan(\theta/2)$$

Start with basic xform

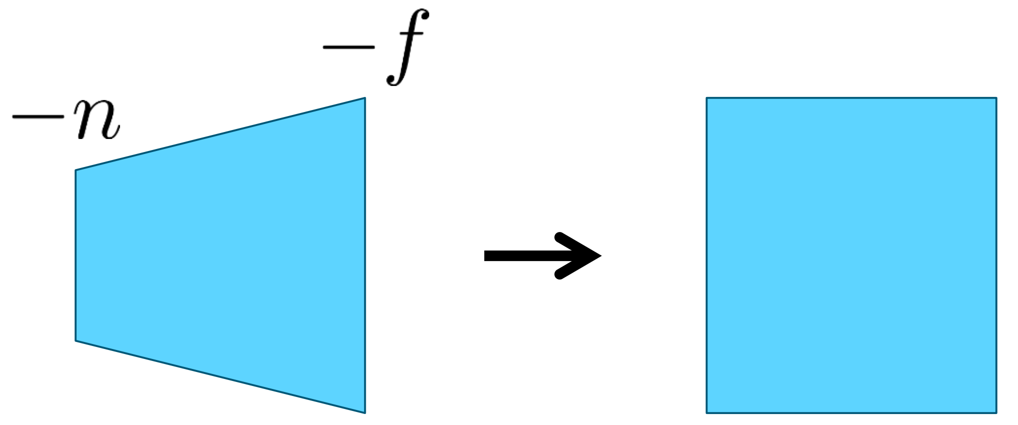
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

then refine



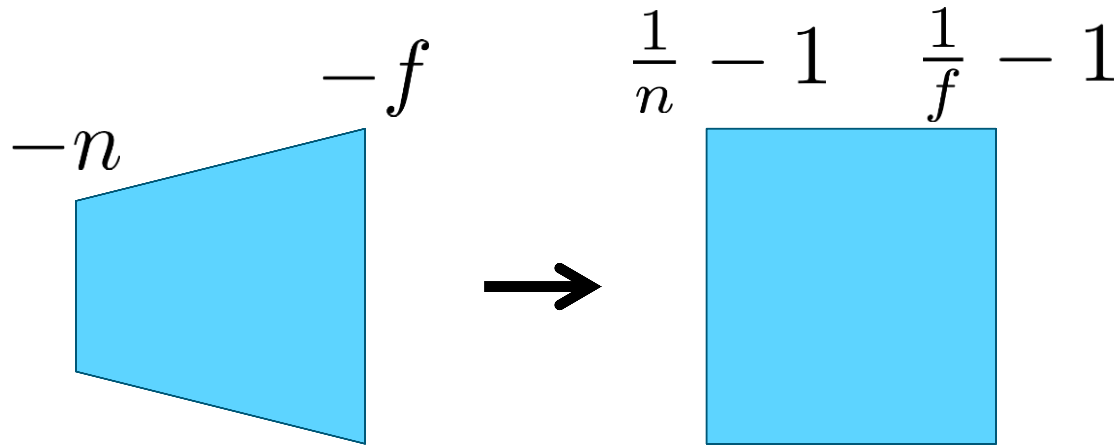
After the Transformation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



After the Transformation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



Needs translation, flip, scaling

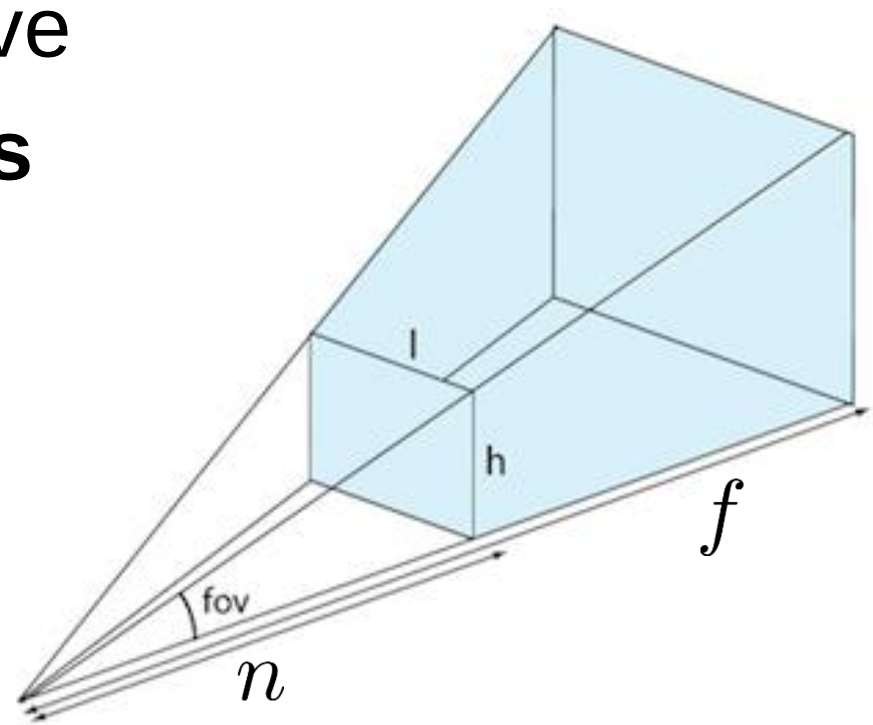
Midpoint: $\frac{1}{2n} + \frac{1}{2f} - 1$ Extent: $\frac{f-n}{fn}$

Perspective Transformation

$$h = 2n\tan(\theta/2)$$
$$l = 2an\tan(\theta/2)$$

Transformations:

- Apply perspective
- **Translate z axis**
- Flip Z
- Scale axes



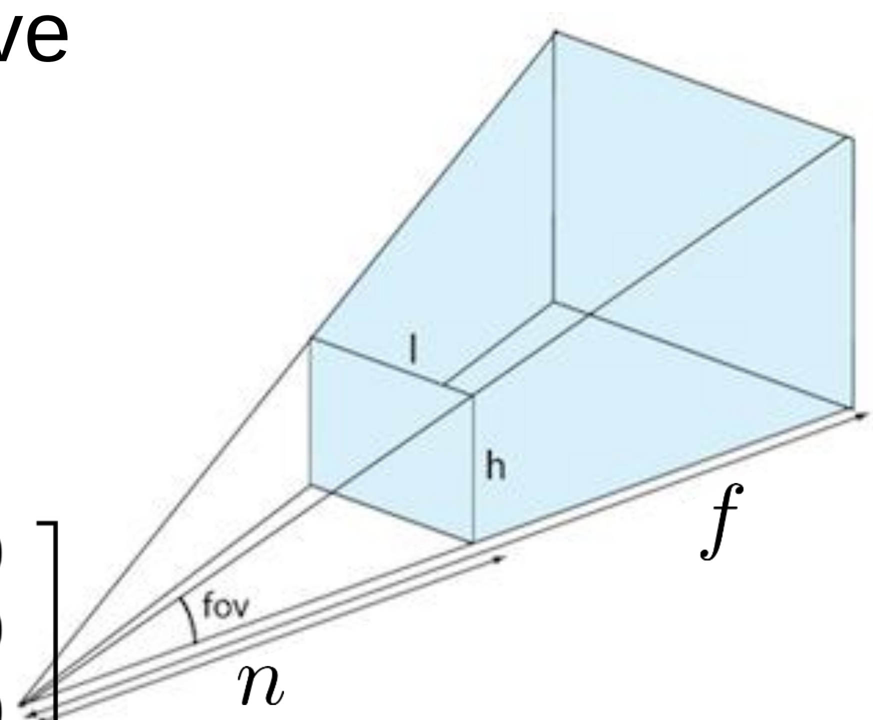
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 - \frac{1}{2f} - \frac{1}{2n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Transformation

$$h = 2n\tan(\theta/2)$$
$$l = 2an\tan(\theta/2)$$

Transformations:

- Apply perspective
- Translate z axis
- **Flip Z**
- **Scale axes**



$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & \frac{-2fn}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Transformation

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & \frac{-2fn}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 - \frac{1}{2f} - \frac{1}{2n} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Perspective Transformation

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & \frac{-2fn}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1-\frac{1}{2f}-\frac{1}{2n} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & \frac{-2fn}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{f+n}{2fn} & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Some Consequences

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Where does point (0,0,-z) go for large z?

Some Consequences

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Where does point (0,0,-z) go for large z?

$$\left[0,\ 0\ ,z\frac{f+n}{f-n}+\frac{-2fn}{f-n},z\right]=\left(0\ ,0\ ,\frac{f+n}{f-n}-\frac{2fn}{z(f-n)}\right)$$

$$\to\left(0\ ,0\ ,\frac{f+n}{f-n}\right)$$ at finite depth!

Some Consequences

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Where does point (0,0,-z) go for large z?

$$\left[0,\ 0\ , z\frac{f+n}{f-n} + \frac{-2fn}{f-n}, z\right] = \left(0\ , 0\ , \frac{f+n}{f-n} - \frac{2fn}{z(f-n)}\right)$$

$$\rightarrow \left(0\ , 0\ , \frac{f+n}{f-n}\right) \text{ at finite depth!}$$

Where do points behind camera (+z) go?

Some Consequences

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Where does point (0,0,-z) go for large z?

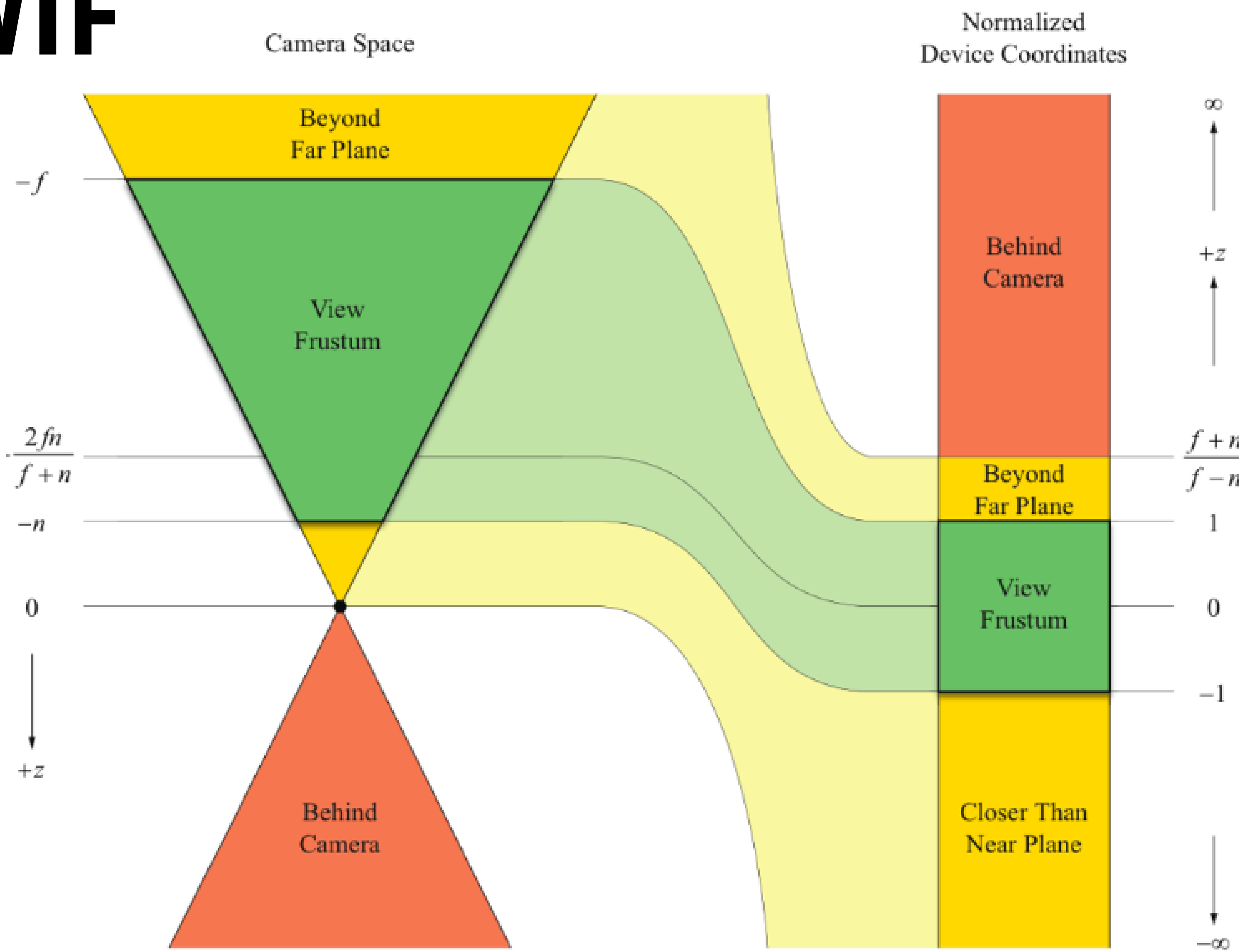
$$\left[0,\ 0\ ,z\frac{f+n}{f-n} + \frac{-2fn}{f-n}, z\right] = \left(0\ ,0\ ,\frac{f+n}{f-n} - \frac{2fn}{z(f-n)}\right)$$

$$\rightarrow \left(0\ ,0\ ,\frac{f+n}{f-n}\right)$$ at finite depth!

Where do points behind camera (+z) go?

$$\left(0\ ,0\ ,\frac{f+n}{f-n} + \frac{2fn}{z(f-n)}\right)$$ positive depth!

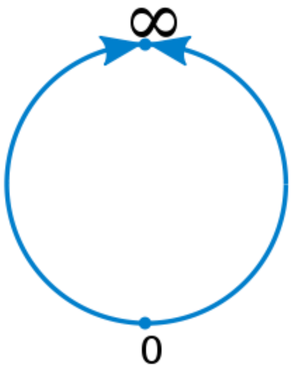
WTF

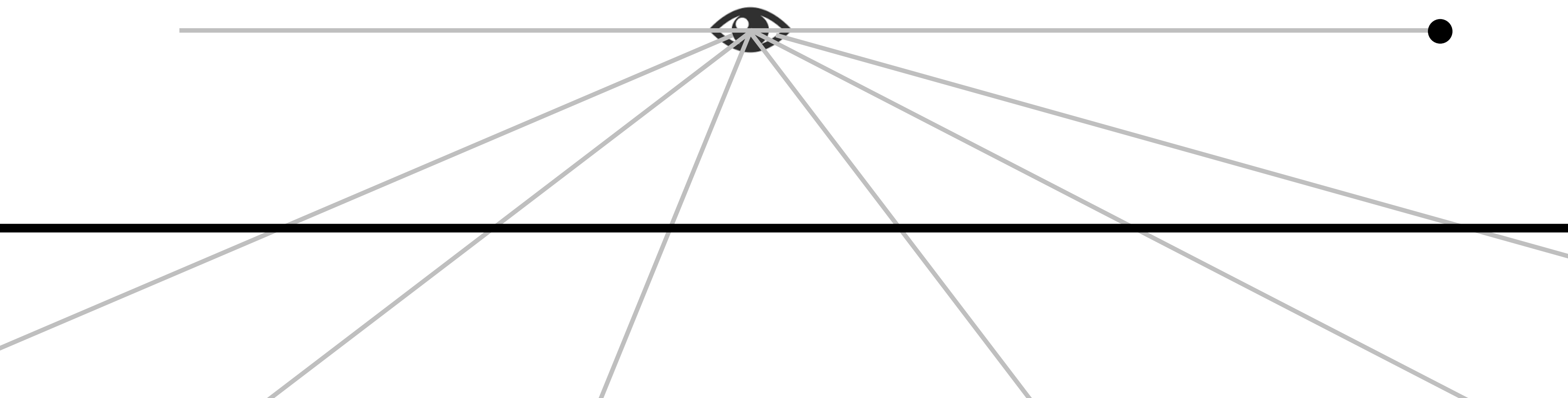


Camera Space
Normalized
Device Coordinates
Beyond
Far Plane
−f
View
Frustum
2fn
f + n
−n
0
+z
Behind
Camera
Behind
Camera
+z
Beyond
Far Plane
View
Frustum
Closer Than
Near Plane
1
0
−1
−∞

Why Did This Happen?

Translation during perspective step
rotated the projective plane





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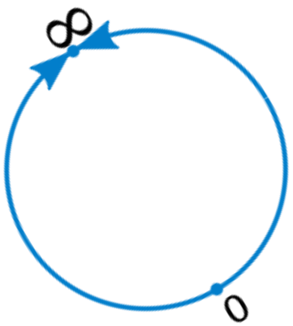
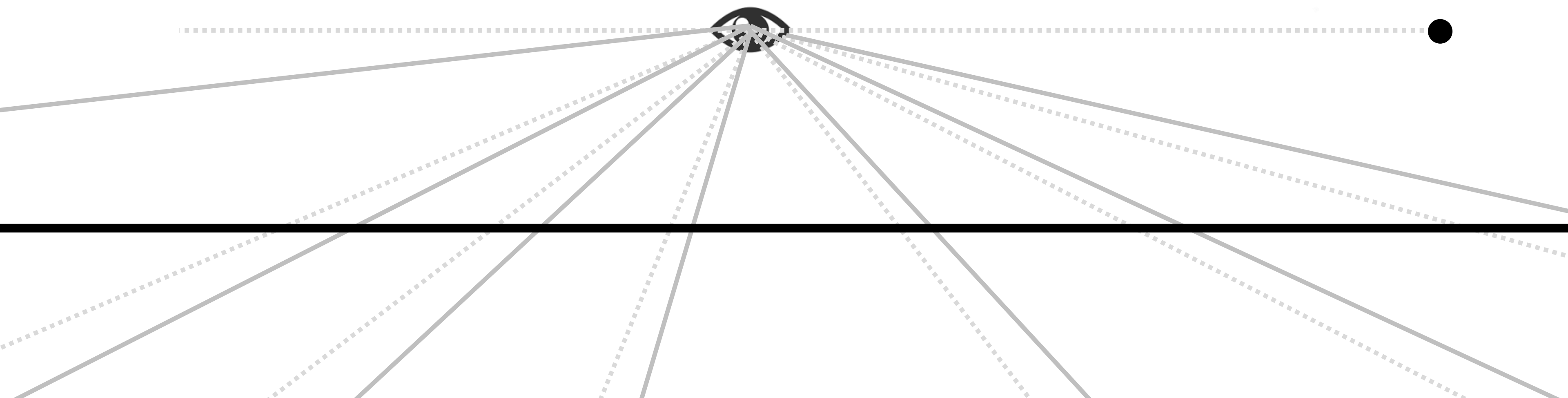
rotated the projective plane





Far Plane At Infinity

Tempting to set far plane at infinity:

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -1 & -2n \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Far Plane At Infinity

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$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -1 & -2n \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Usually **bad** idea: depth buffer loses all precision

Near Plane At Zero

Tempting to set near plane at zero:

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Near Plane At Zero

Tempting to set near plane at zero:

$$\begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\cot(\theta/2)}{a} & 0 & 0 & 0 \\ 0 & \cot(\theta/2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Usually **bad** idea: all depths set to 1