CS 395T
Lecture 7: Two-View Geometry

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Uncalibrated Camera – Intrinsic Parameters are unknown

\[ x' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = Kx = \begin{bmatrix} f s_x & f s_\theta & o_x \\ 0 & f s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \]

Linear transformation

pixel coordinates

(0, 0)

\( x' \)

\( y' \)

(\( o_x, o_y \))

calibrated coordinates

\( K \)

\( z \)

\( O \)

\( x \)

\( y \)

\( s_x \)

\( s_y \)
Overview

- Calibration with a rig (Checkborad for example)
- Uncalibrated epipolar geometry
- Ambiguities in image formation
- Stratified reconstruction
Uncalibrated Camera Using Homogeneous Coordinates

\[ \mathbf{X} = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1) \]

Last Lecture:
- Image plane coordinates \( \mathbf{x} = [x, y, 1]^T \)
- Camera extrinsic parameters \( \mathbf{g} = (R, T) \)
- Perspective projection \( \lambda \mathbf{x} = [R, T] \mathbf{X} \)

This Lecture:
- Pixel coordinates \( \mathbf{x}' = K \mathbf{x} \)
- Projection matrix \( \lambda \mathbf{x}' = \Pi \mathbf{X} = [KR, KT] \mathbf{X} \)
Calibration with a Rig

Use the fact that both 3-D and 2-D coordinates of feature points on a pre-fabricated object (e.g., a cube) are known.
Calibration with a Rig

• Given 3-D coordinates on known object $X$

$$\lambda x' = [KR, KT]X \quad \rightarrow \quad \lambda x' = \Pi X$$

$$\lambda \begin{bmatrix} x^i \\ y^i \\ 1 \end{bmatrix} = \begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix} \begin{bmatrix} X^i \\ Y^i \\ Z^i \\ 1 \end{bmatrix}$$

• Eliminate unknown scales

$$x^i(\pi_3^T X) = \pi_1^TX,$$

$$y^i(\pi_3^T X) = \pi_2^TX$$
Calibration with a Rig

- Recover projection matrix $\Pi = [KR, KT] = [R', T']$

  $\Pi^s = [\pi_{11}, \pi_{21}, \pi_{31}, \pi_{12}, \pi_{22}, \pi_{32}, \pi_{13}, \pi_{23}, \pi_{33}, \pi_{14}, \pi_{24}, \pi_{34}]^T$

  \[ \min ||MP^s||^2 \quad \text{subject to} \quad ||P^s||^2 = 1 \]

  Again singular value decomposition

- Factor the $KR$ into $R \in SO(3)$ and $K$ using QR decomposition

- Solve for translation $T = K^{-1}T'$
Uncalibrated Epipolar Geometry

\[ \lambda_2 K x_2 = KR \lambda_1 x_1 + KT \]

\[ \lambda_2 x'_2 = KRK^{-1} \lambda_1 x'_1 + T' \]

- Epipolar constraint
- Fundamental matrix
- Equivalent forms of

\[ x'^T_2 K^{-T} \hat{T} RK^{-1} x'_1 = 0 \]

\[ F = K^{-T} \hat{T} RK^{-1} \]

\[ F = K^{-T} \hat{T} RK^{-1} = \hat{T}' KRK^{-1} \]
Properties of the Fundamental Matrix

- Epipolar lines \( l_1, l_2 \)
- Epipoles \( e_1, e_2 \)

\[
x'^T F x' = 0
\]

\[
l_1 \sim F^T x'_2
\]

\[
F e_1 = 0
\]

\[
l'^T_i x'_i = 0
\]

\[
l'^T_i e_i = 0
\]

\[
l_2 \sim F x'_1
\]

\[
e_2^T F = 0
\]
Properties of the Fundamental Matrix

**Remark 6.1.** Characterization of the fundamental matrix. A non-zero matrix \( F \in \mathbb{R}^{3 \times 3} \) is a fundamental matrix if \( F \) has a singular value decomposition (SVD): \( E = U\Sigma V^T \) with

\[
\Sigma = \text{diag}\{\sigma_1, \sigma_2, 0\}
\]

for some \( \sigma_1, \sigma_2 \in \mathbb{R}_+ \).

There is little structure in the matrix \( F \) except that

\[
\det(F) = 0
\]
Estimating Fundamental Matrix

• Find such F that the epipolar error is minimized

\[ \min_F \sum_{j=1}^{n} (\mathbf{x}_{2,j}'^T F \mathbf{x}_{1,j}')^2 \quad \text{s.t.} \quad \|F\|_F^2 = 1 \]

• Fundamental matrix can be estimated up to scale

• Denote \( \mathbf{a} = \mathbf{x}_1' \otimes \mathbf{x}_2' \)

\[ \mathbf{a} = [x_1x_2, x_1y_2, x_1z_2, y_1x_2, y_1y_2, y_1z_2, z_1x_2, z_1y_2, z_1z_2]^T \]

\[ F^s = [f_1, f_4, f_7, f_2, f_5, f_8, f_3, f_6, f_9]^T \]

• Rewrite \( \mathbf{a}^T F^s = 0 \)

\[ \min_{F^s} \|AF^s\|^2 \quad \text{s.t.} \quad \|F^s\|^2 = 1 \]
Two view linear algorithm – 8-point algorithm

- Solve the LLSE problem:
  \[
  \min_F \sum_{j=1}^{n} (\mathbf{x}_{2,j}^T F \mathbf{x}_{1,j}^{'})^2 \quad s.t. \quad \|F\|^2_F = 1
  \]

- Solution eigenvector associated with smallest eigenvalue of $A^T A$

- Compute SVD of F recovered from data
  \[
  F = U \Sigma V^T \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3)
  \]

- Project onto the essential manifold:
  \[
  \Sigma' = \text{diag}(\sigma_1, \sigma_2, 0) \quad F = U \Sigma' V^T
  \]

- $F$ cannot be unambiguously decomposed into pose and calibration
  \[
  F = K^{-T} \hat{R} K^{-1}
  \]
What Does $F$ Tell Us?

• $F$ can be inferred from point matches (eight-point algorithm)

• Cannot extract motion, structure and calibration from one fundamental matrix (two views)

• $F$ allows reconstruction up to a projective transformation (as we will see soon)

• $F$ encodes all the geometric information among two views when no additional information is available
Decomposing the Fundamental Matrix

\[ F = K^{-T} \hat{T} RK^{-1} = \hat{T}' KRK^{-1} \]

- Decomposition of the fundamental matrix into a skew-symmetric matrix and a nonsingular matrix
  \[ F \rightarrow \Pi = [R', T'] \quad \Rightarrow \quad F = \hat{T}' R'. \]
- Decomposition of F is not unique
  \[ x'_2 \hat{T}'(T'v^T + KRK^{-1})x'_1 = 0 \quad T' = KT \]
- Unknown parameters - ambiguity
  \[ v = [v_1, v_2, v_3]^T \in \mathbb{R}^3, \quad v_4 \in \mathbb{R} \]
- Corresponding projection matrix
  \[ \Pi = [KRK^{-1} + T'v^T, v_4T'] \]
Projective Reconstruction

- From image correspondences, extract $F$, followed by computation of projection matrices $\Pi_{ip}$ and structure $X_p$
- Canonical decomposition
  \[ F \quad \mapsto \quad \Pi_{1p} = [I, 0], \quad \Pi_{2p} = [(\hat{T}')^T F, T'] \]
- Given projection matrices --- recover structure $X_p$
  \[
  \lambda_1 x'_1 = \Pi_{1p} X_p = [I, 0] X_p, \\
  \lambda_2 x'_2 = \Pi_{2p} X_p = [(\hat{T}')^T F, T'] X_p.
  \]
- Projective ambiguity --- non-singular 4x4 matrix
  \[
  \lambda_i x'_i = \Pi_{ip} H^{-1} H X_p \\
  \lambda_i x'_i = \tilde{\Pi}_{1p} \tilde{X}_p
  \]

Both $\Pi_{ip}$ and $\tilde{\Pi}_{ip}$ are consistent with the epipolar geometry – give the same fundamental matrix
Projective Reconstruction

- Given projection matrices recover projective structure

\[
(x_1 \pi_1^{3T})X_p = \pi_1^{1T}X_p, \quad (y_1 \pi_1^{3T})X_p = \pi_1^{2T}X_p,
\]
\[
(x_2 \pi_2^{3T})X_p = \pi_2^{1T}X_p, \quad (y_2 \pi_2^{3T})X_p = \pi_2^{2T}X_p,
\]

- This is a linear problem and can be solve using linear least-squares

\[
MX_p = 0
\]

- Projective reconstruction – projective camera matrices and projective structure

\[
X_e = HX_p
\]
Euclidean vs Projective reconstruction

- **Euclidean reconstruction** – true metric properties of objects lengths (distances), angles, parallelism are preserved
- Unchanged under rigid body transformations

- **Projective reconstruction** – lengths, angles, parallelism are NOT preserved – we get distorted images of objects – their distorted 3D counterparts --> 3D projective reconstruction
Ambiguities in Image Formation

• Structure of the (uncalibrated) projection matrix

\[ \lambda x' = \Pi X = (\Pi H^{-1})(HX) = \tilde{\Pi} \tilde{X} \quad \Pi = [KR, KT] \]

• For any invertible 4x4 matrix H

• In the uncalibrated case we cannot distinguish between \( \Pi \) camera imaging word X from camera \( \tilde{\Pi} \) imaging distorted world \( \tilde{X} \)

• In general, \( H \) is of the following form

\[ H^{-1} = \begin{bmatrix} G & b \\ v^T & v_4 \end{bmatrix} \]

• In order to preserve the choice of the first reference frame we can restrict some DOF of \( H \)
Structure of the Projective Ambiguity

- 1\textsuperscript{st} frame as reference
  \[ \lambda_1 x'_1 = K_1 \Pi_0 X_e \]
  \[ \lambda_1 x'_1 = K_1 \Pi_0 H^{-1} HX_e = \Pi_{1p} X_p \]

- Choose the projective reference frame

\[ \Pi_{1p} = [I_{3 \times 3}, 0] \] then ambiguity is

\[ H^{-1} = \begin{bmatrix} K^{-1}_1 & 0 \\ v^T & v_4 \end{bmatrix} \]

\( H^{-1} \) can be further decomposed as

\[ H^{-1} = \begin{bmatrix} K^{-1}_1 & 0 \\ v^T & v_4 \end{bmatrix} = \begin{bmatrix} K^{-1}_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} = H_{a}^{-1} H_{p}^{-1} \]
Stratified (Euclidean) Reconstruction

- General ambiguity – while preserving choice of first reference frame

\[ H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} \]

- Decomposing the ambiguity into affine and projective one

\[ H^{-1} = H_a^{-1} H_p^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ v^T & v_4 \end{bmatrix} \]

- Note the different effect of the 4-th homogeneous coordinate
Will continue next lecture