GAMES
Surface Reconstruction

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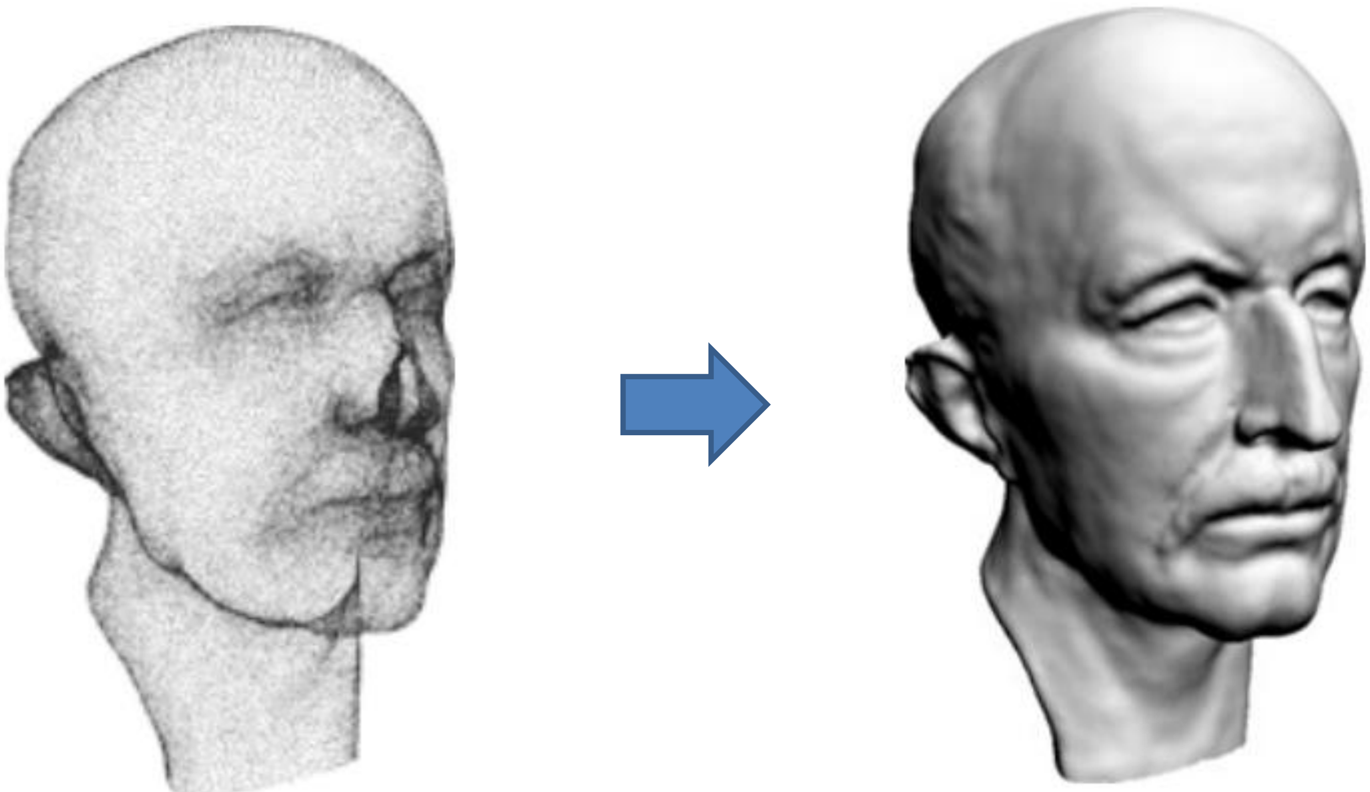
July 23th 2021

Goal



Captured point cloud

Reconstructed model

Two approaches

- Implicit
  - Signed distance function estimation
  - Mesh approximation
  - Fast and efficient
- Explicit
  - Local surface connectivity estimation
  - Computation geometry based

Implicit-Based Methods

Surface Reconstruction from Unorganized Points

[H. Hoppe, T. DeRose, T. Duchamp, J. McDonald, W. Stuetzle
SIGGRAPH 1992]

Method Pipeline

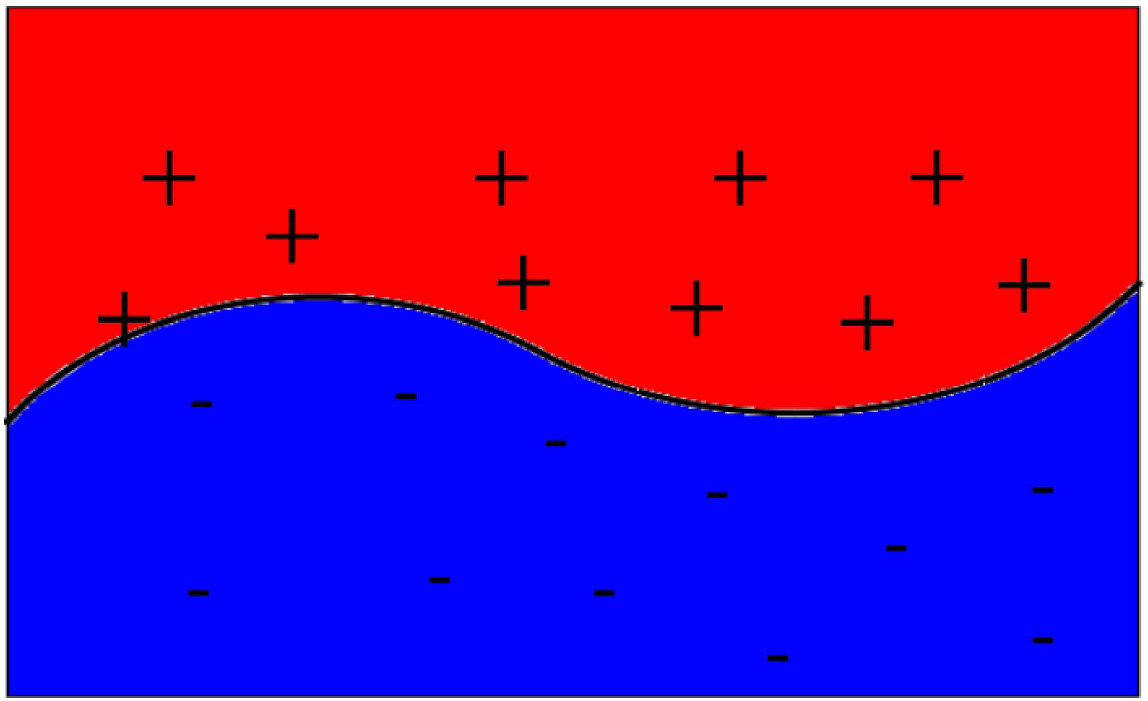
- Input:
  - Cloud of points
  - Orientation not required

- Output:
  - Triangular mesh
  - Possible boundary edges

- Guarantees:
  - Manifold

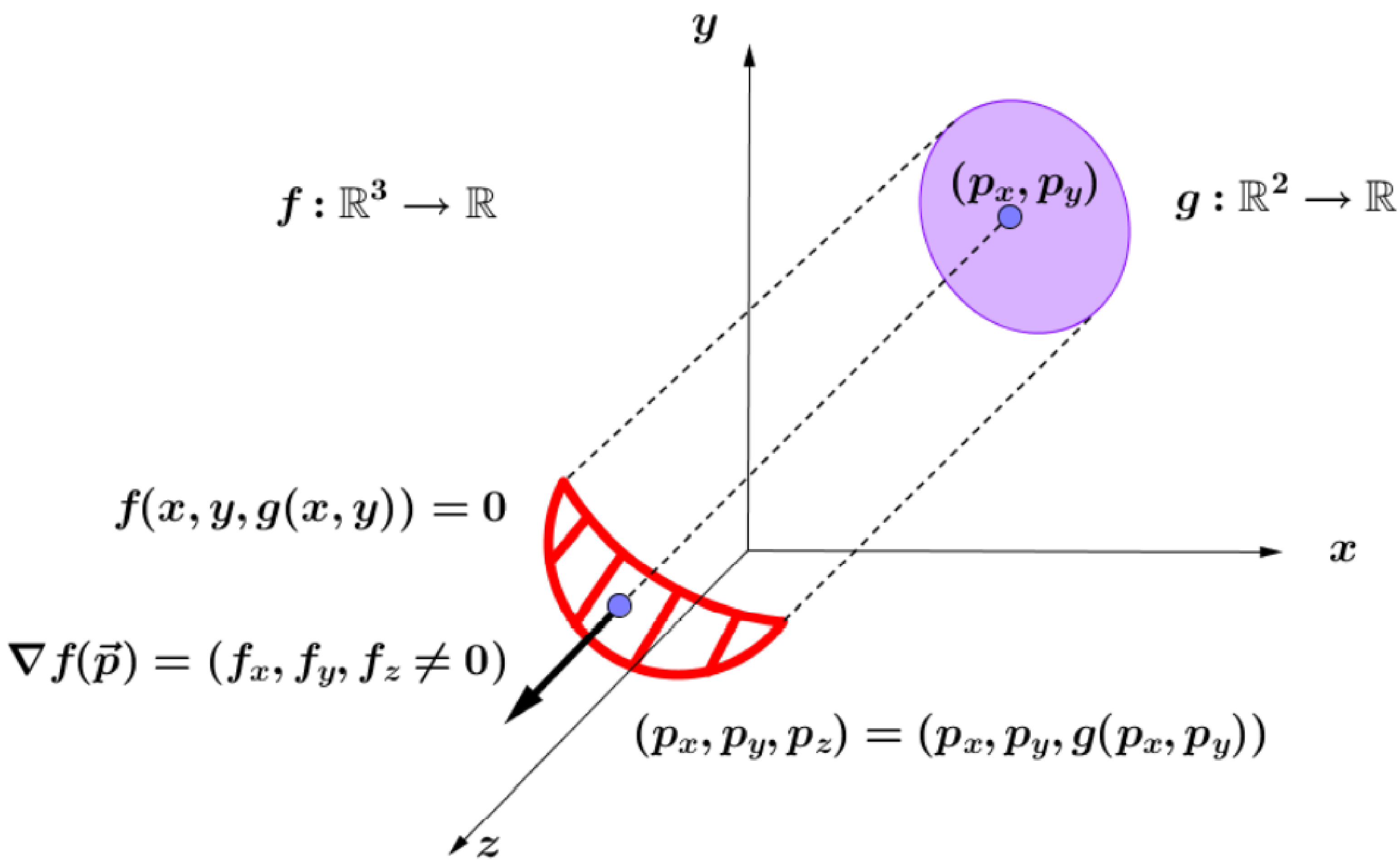
Implicit Surfaces : Regular value

- We would like to represent a function as the zero set of a function $f: R^3 \to R$.



- We say that zero is a regular value of f if $\nabla f(p) \neq 0$ for all points such that $f(p) = 0$.

Implicit Function Theorem



Computation of the Signed Distance Function

- **For each sample fit a tangent plane using its k-Nearest Neighbours**

- Define a coherent orientation for the tangent plane of all sample points

- For any $p \in R^3$ the signed distance function is given by its closest (oriented) tangent plane.

Tangent Plane Fitting

Find a plane that fits, in the Least Squares sense, to its K- Nearest Neighbours:

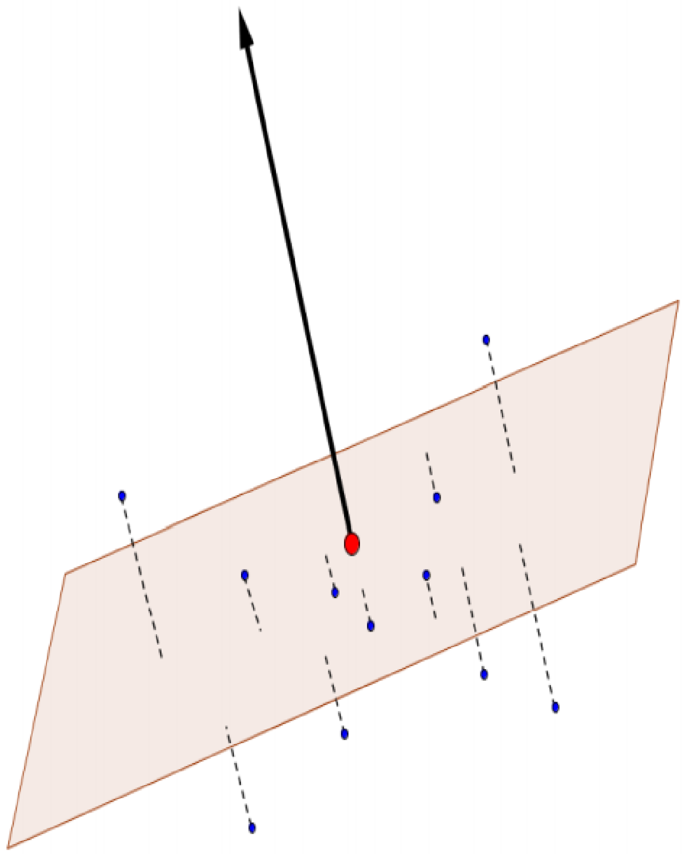
$$\min_{\vec{n}\in S^2, a\in R} \sum_{i=1}^{k} (x_i \cdot \vec{n} - a)^2$$

Properties:

1) This plane passes trhough the baricenter of the neighbours $o = \frac{1}{k}\sum_{i=1}^{k} x_i$

2) The normal direction is given an eigenvector of smallest eigenvalue of the covariance matrix

$$\sum_{i=1}^{k} (x_i - o)(x_i - o)^T$$



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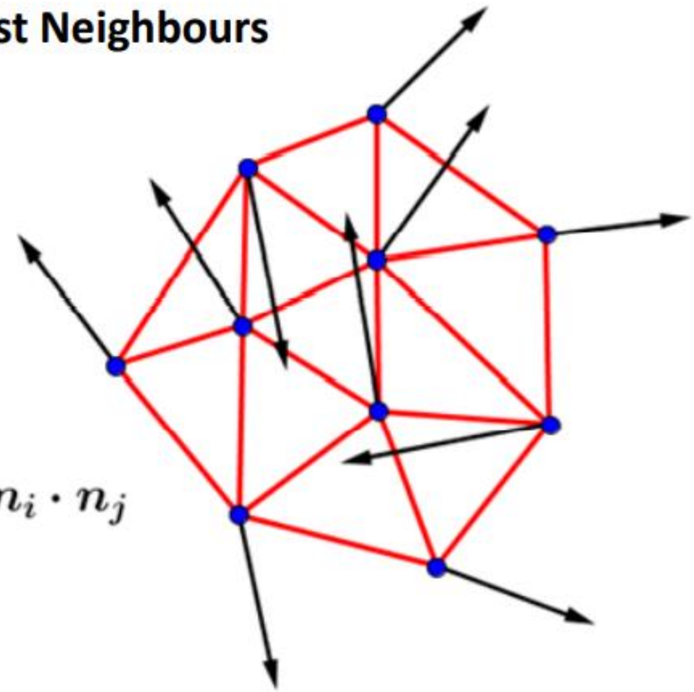
Global Optimization

Riemannian Graph

K- Nearest Neighbours



n_i

$w_{ij} = n_i \cdot n_j$

n_j

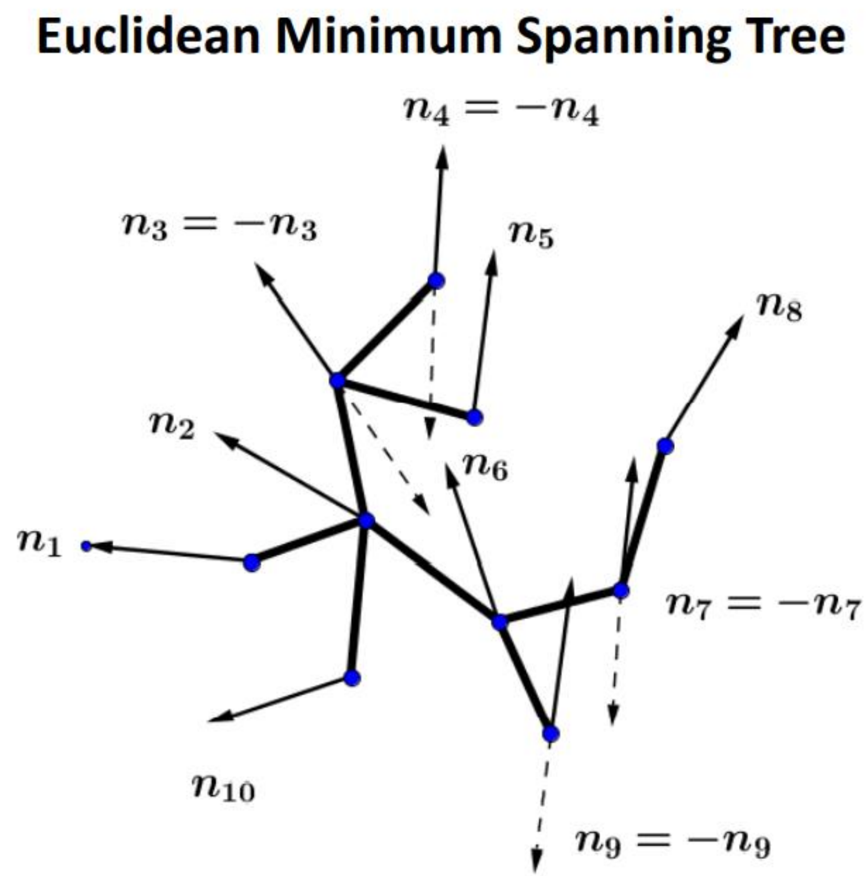
Energy Function

$$\Phi : V \rightarrow \{-1, 1\}$$

$$\max \sum_{(i,j) \in E} w_{ij} \Phi_i \Phi_j$$

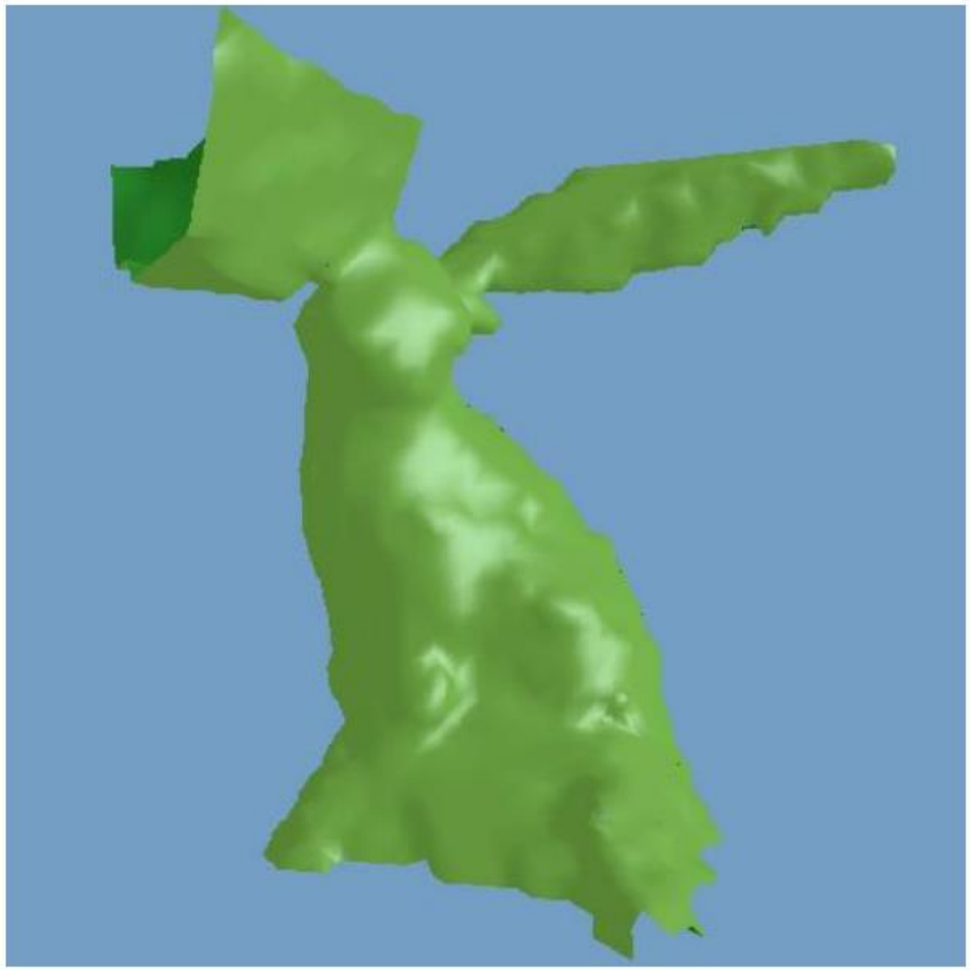
NP-hard!

Normal Propagation By Geometric Proximity



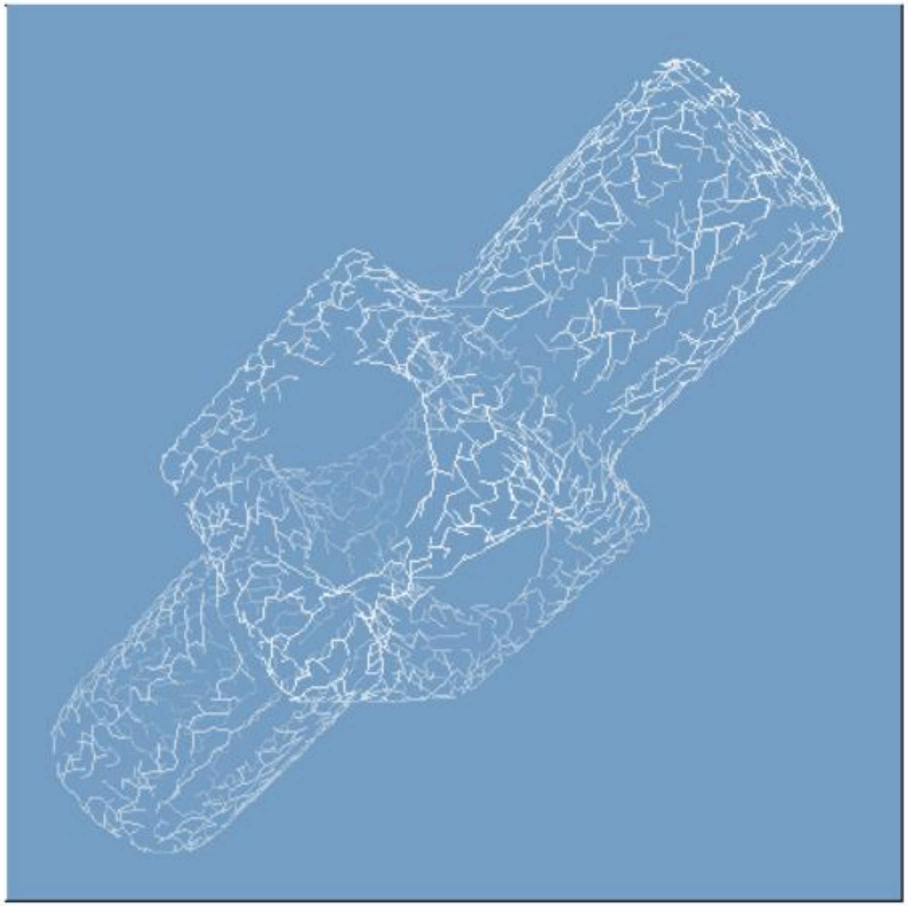
Take $n_j = -n_j$ if $n_i \cdot n_j < 0$

Normal Propagation By Geometric Proximity

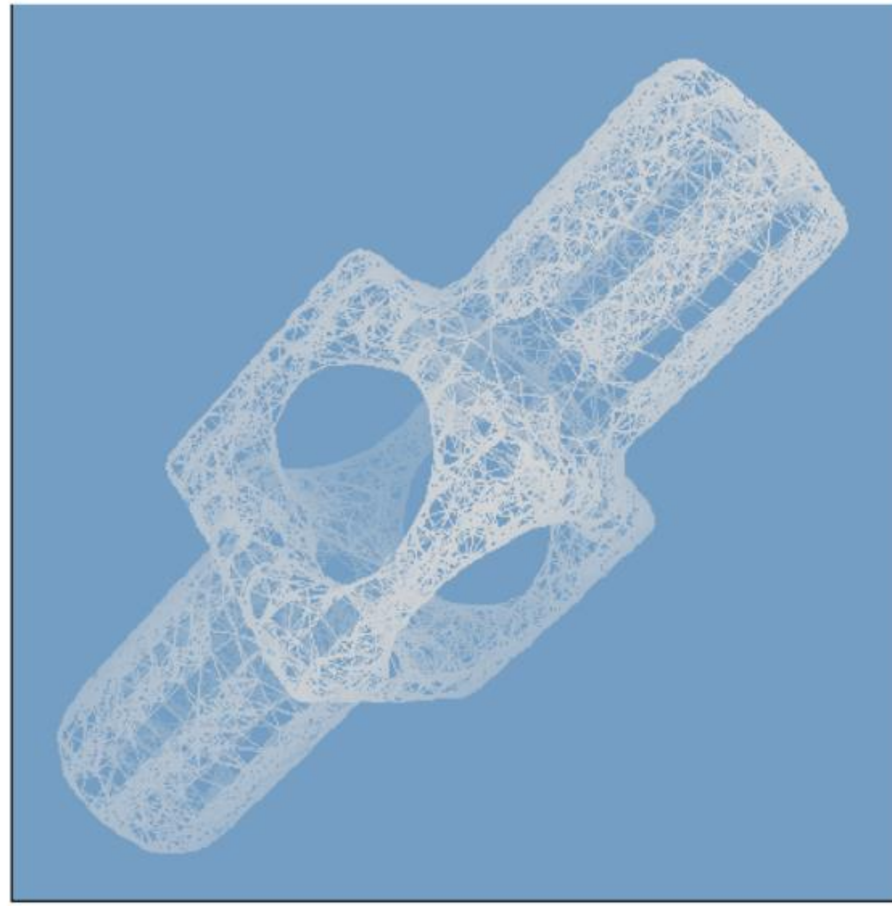


Geometric proximity is not a good criteria for normal propagation.

EMST



Riemmanian Graph

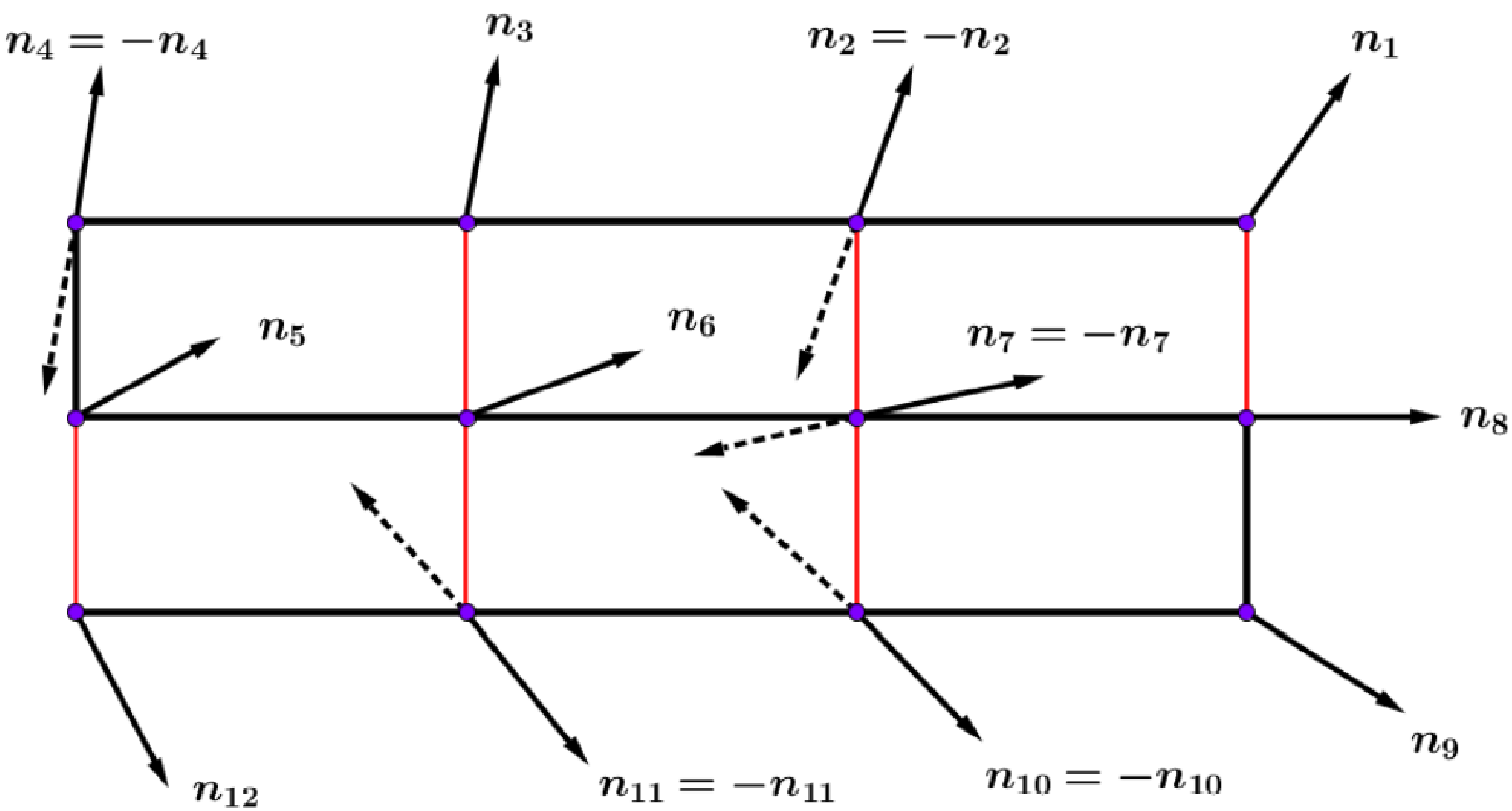


EMST is connected but not dense enough in edges.

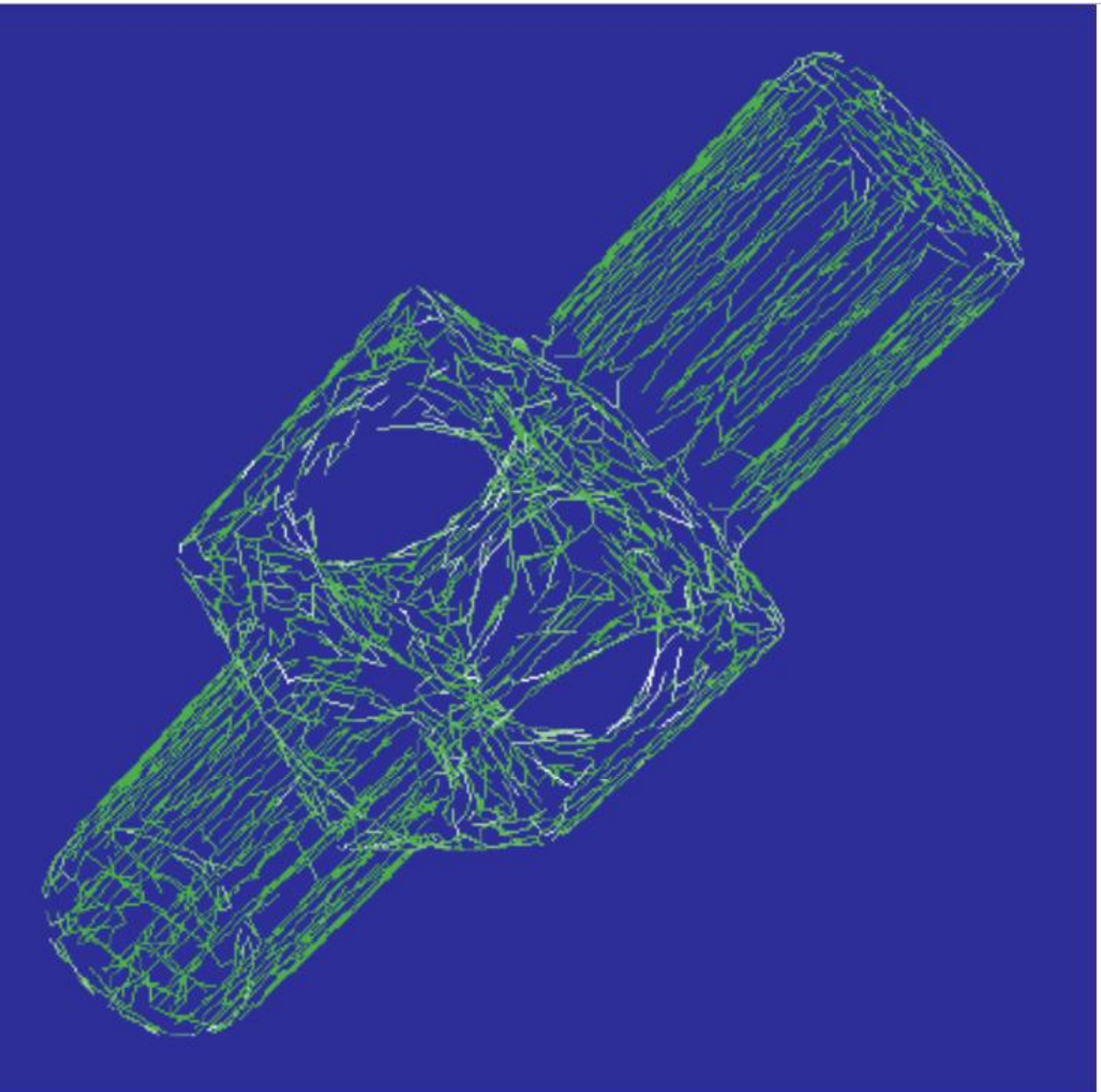
Normal Propagation by Plane Parallelism

1) Construct a Riemmanian graph over the plane centers (o_i's) and edge weights $w_{ij} = 1 - |n_i \cdot n_j|$.

2) Propagate normals along the Minimun Spanning Tree of this graph.

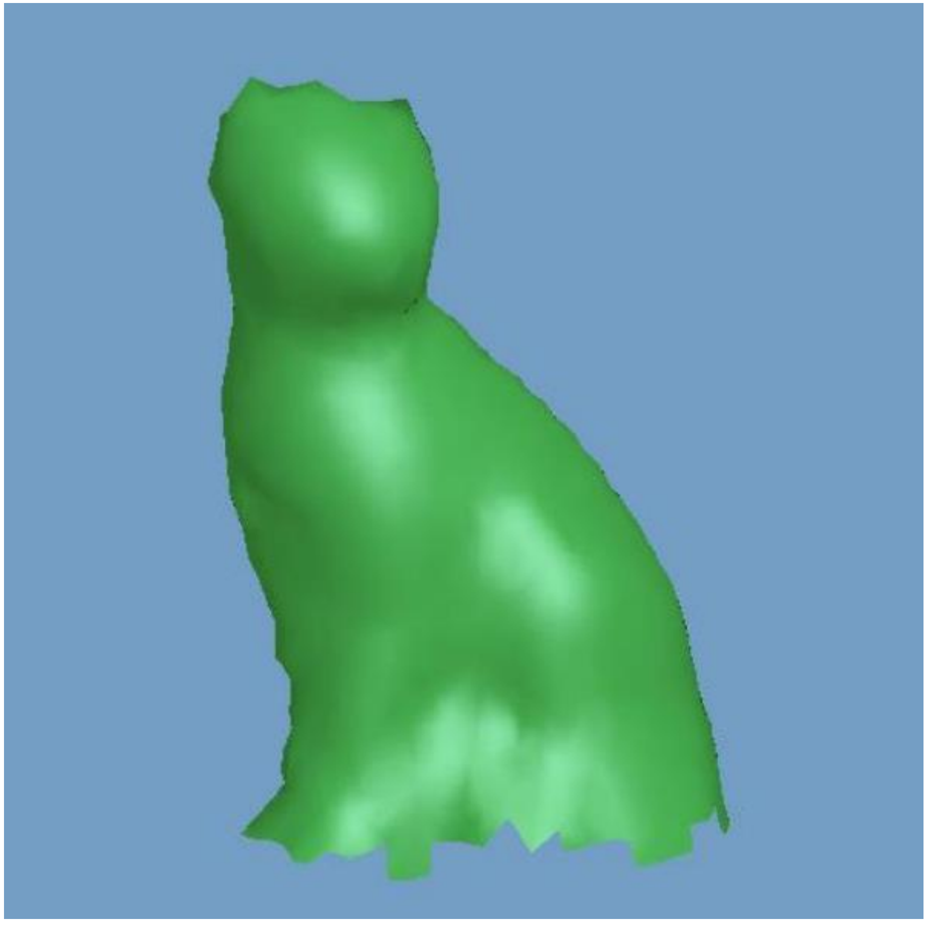


Normal Propagation by Plane Parallelism



Favorates normal proagation along low curvature regions.

Normal Propagation by Plane Parallelism



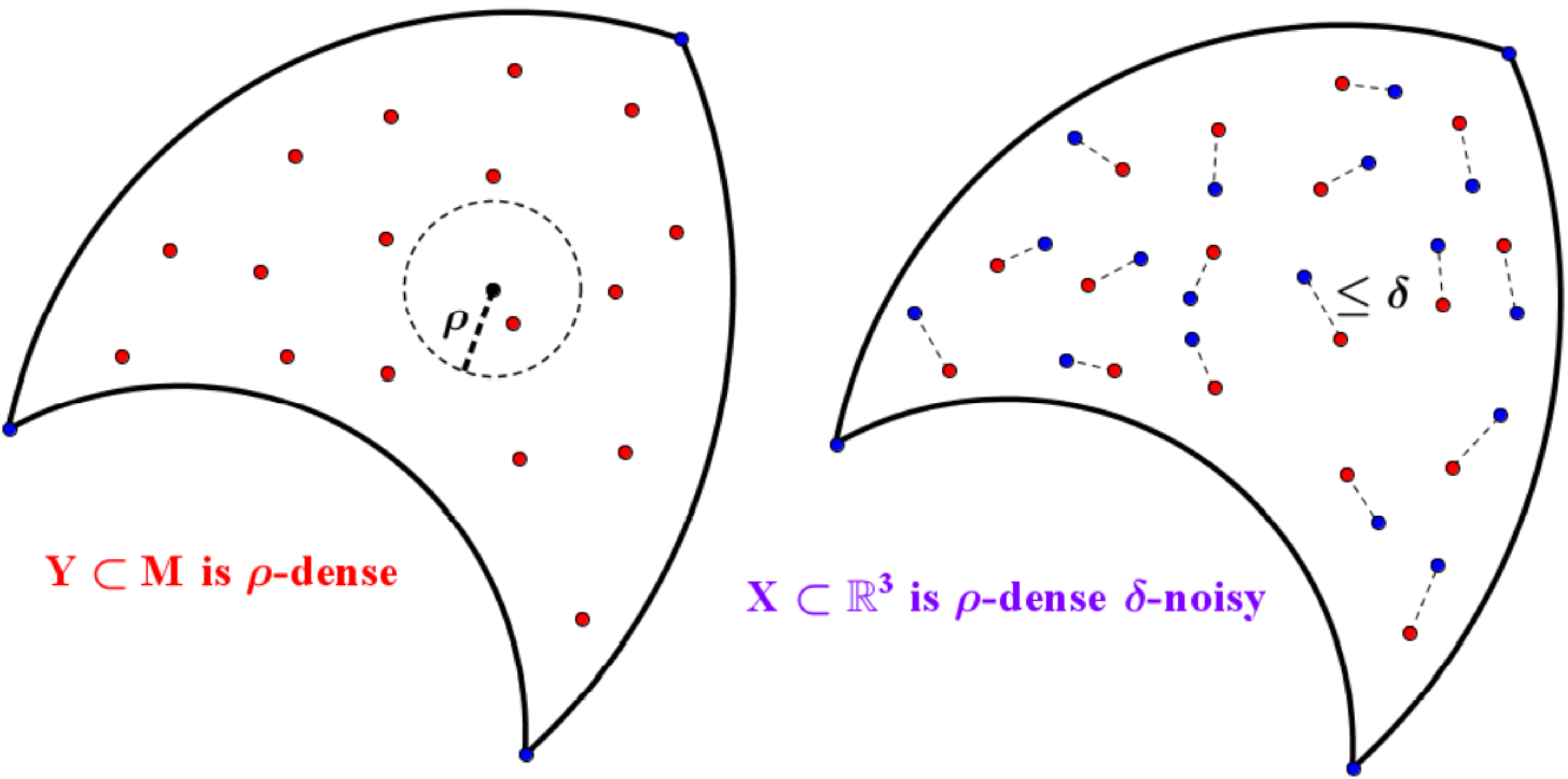
Computation of the Signed Distance Function

- For each sample fit a tangent plane using its k-Nearest Neighbours

- Define a coherent orientation for the tangent plane of all sample points

- **For any $p \in R^3$ the signed distance function is given by its closest (oriented) tangent plane**

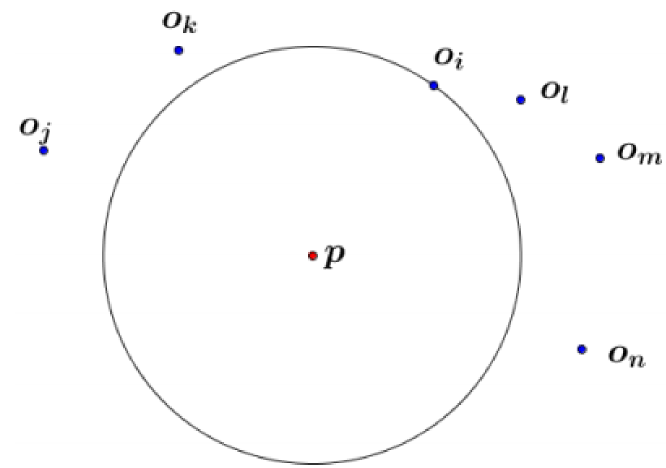
Sampling Assumptions



For any point in the surface the closest sample point in **X** is at most $\rho + \delta$ apart.

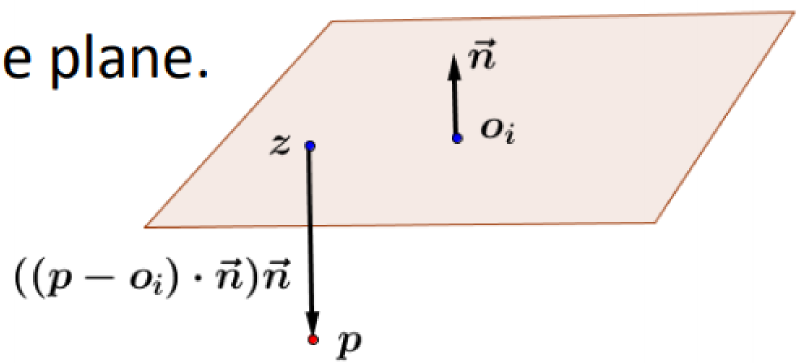
Signed Distance Function

Compute $f(p)$:



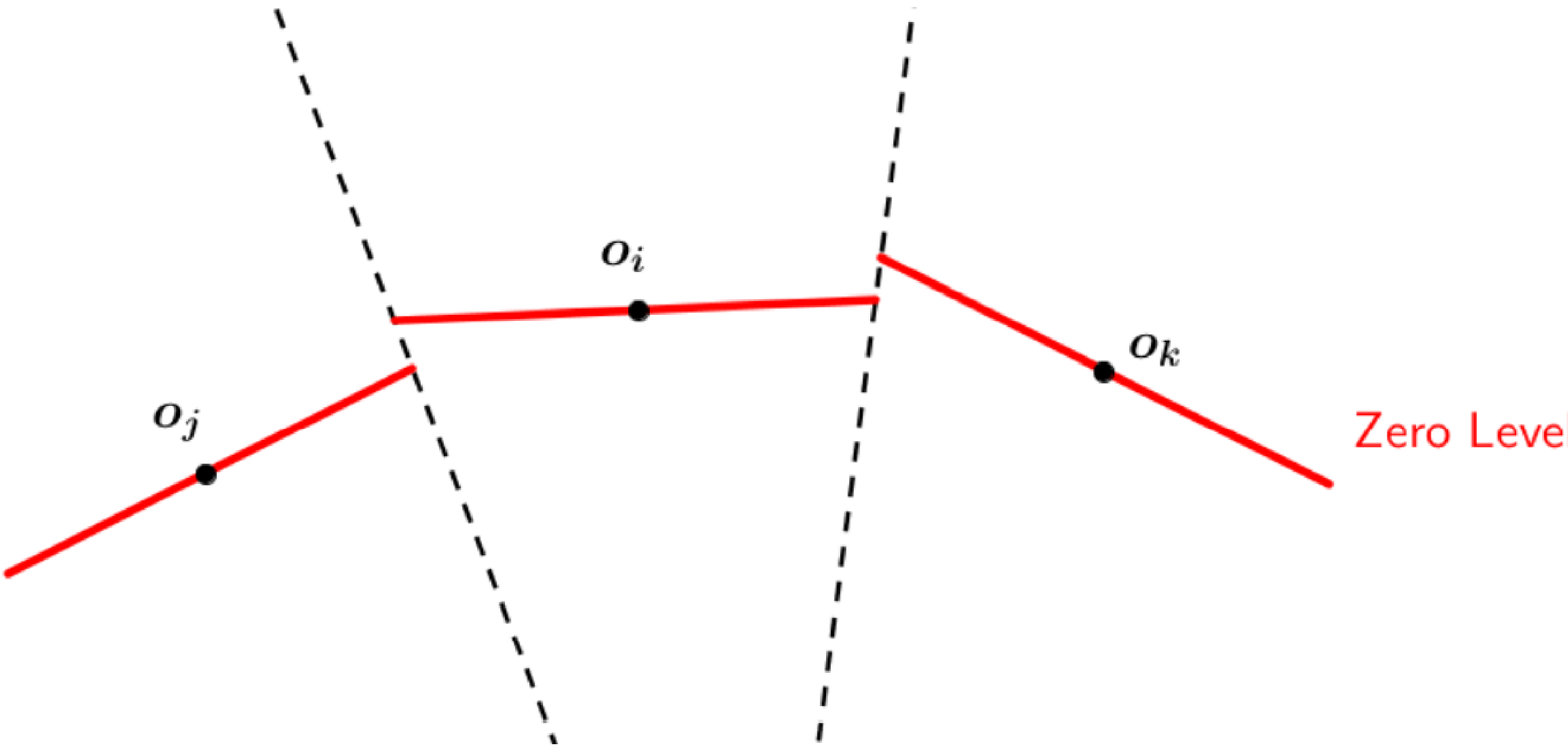
1) Find the closest center to p.

2) Computed the signed distance to the plane.



3) If $d(z, X) < \rho + \delta$ then $f(p) = ((p - o_i) \cdot \vec{n})\vec{n}$.
Otherwise, $f(p)$ is undefined.

Remark



Still it provides a good approximation to reconstruct the surface.

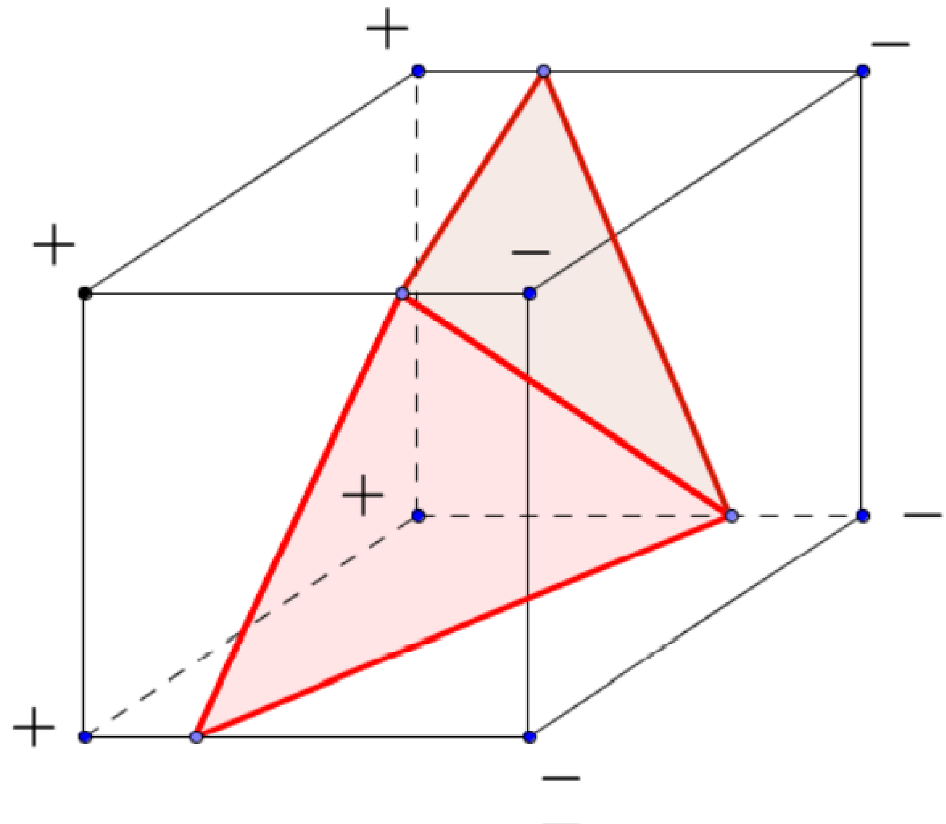
Surface Reconstruction: Marching Cubes

1) Define a cube partition of the space. The edge of each cube should be less than $\rho + \delta$.

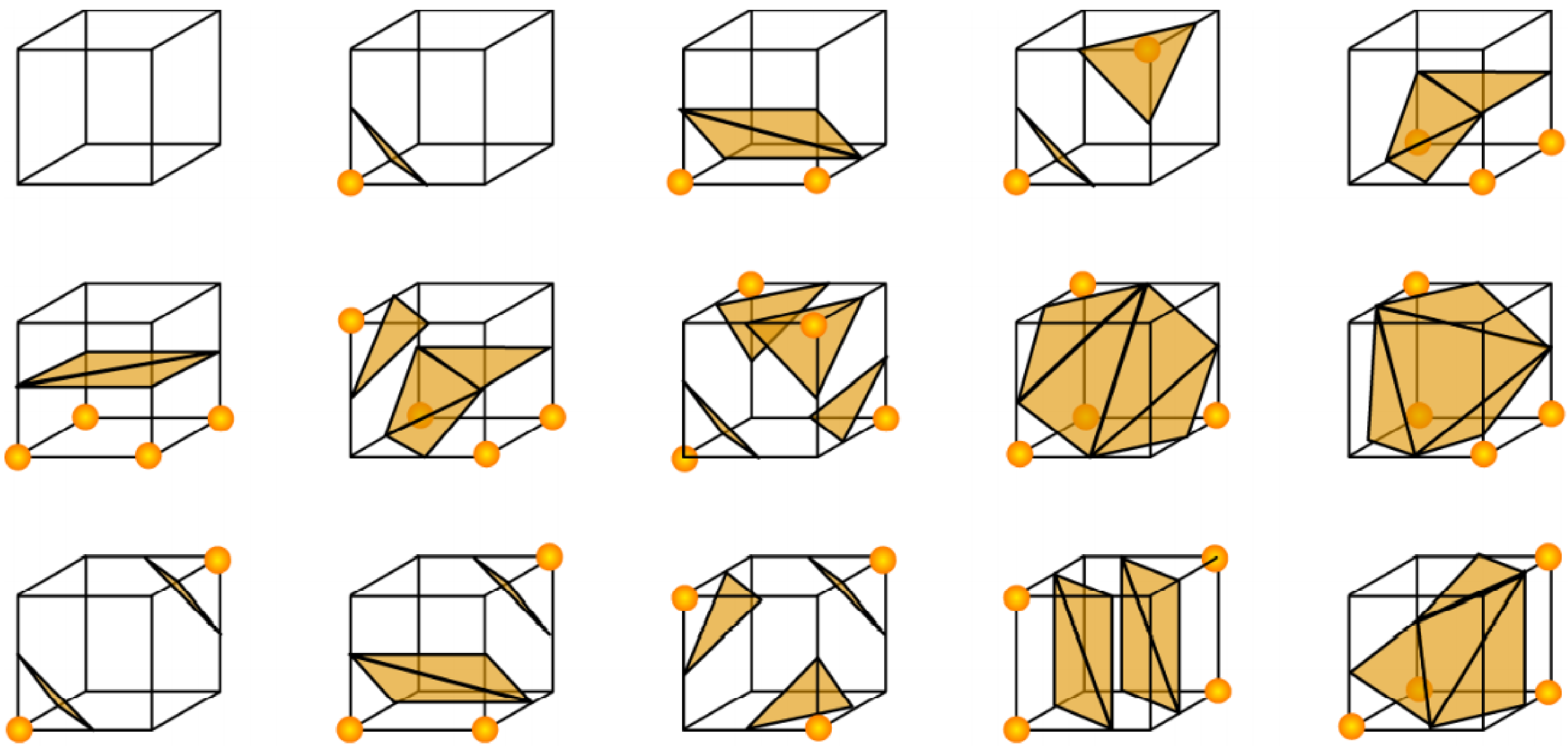
2) Compute the signed distance function on the cube vertices

3) Interpolate zero values (i.e., surface intersections) at changing sign edges.

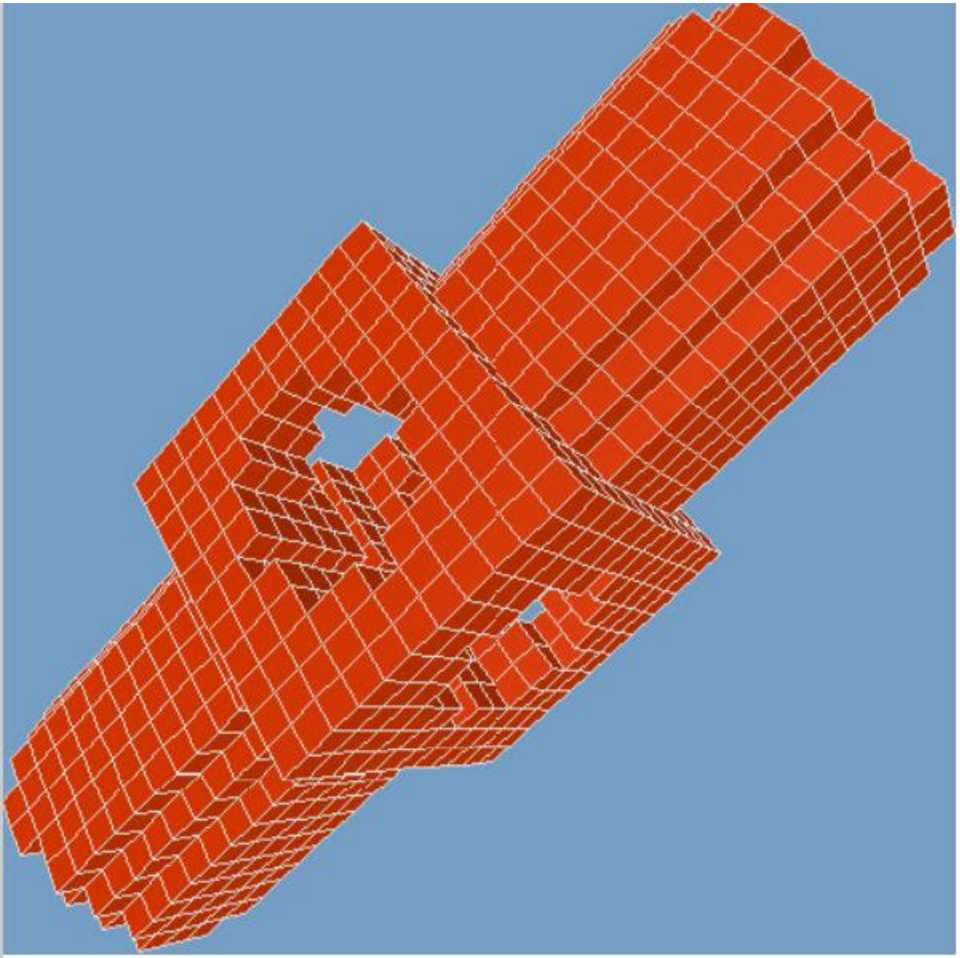
4) Find a trinagulation with vertices at zero values.



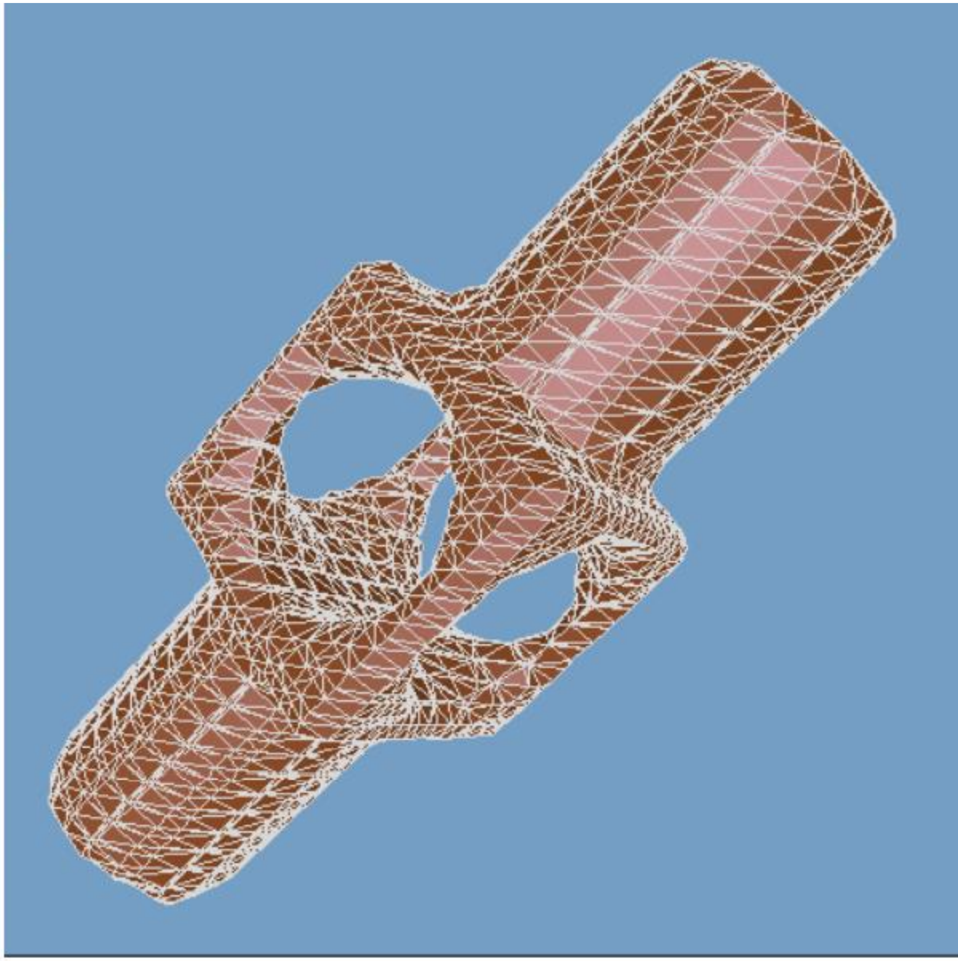
Surface Reconstruction: Marching Cubes



Surface Reconstruction: Marching Cubes



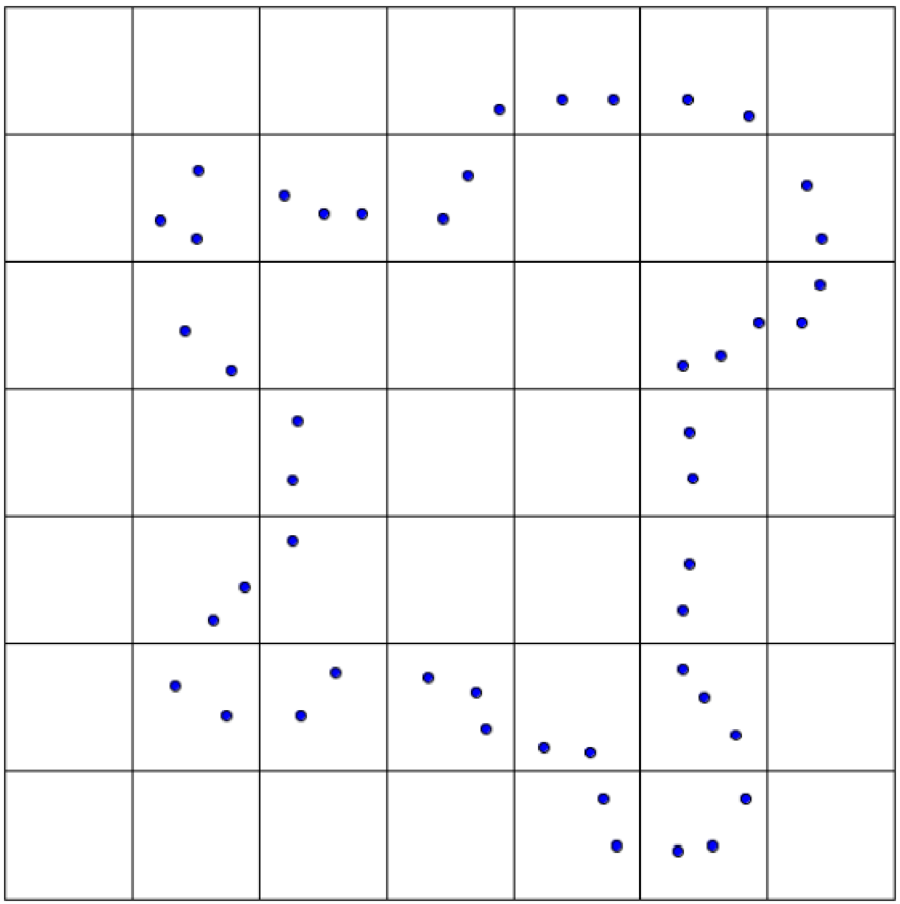
Cubes visited



Triangle Mesh

Data Structure

Group samples by voxels



Constant number of samples per voxel if sampling is unifom.

Complexities:

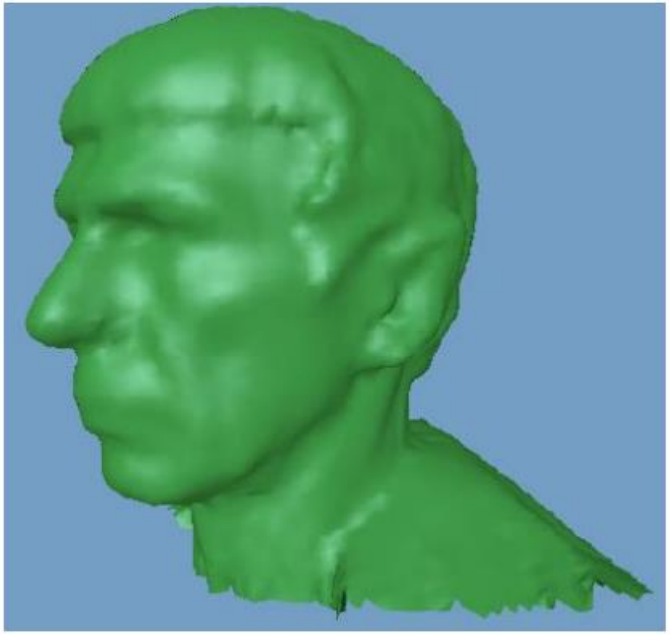
1) Riemmanian Graph construction: $O(nk)$

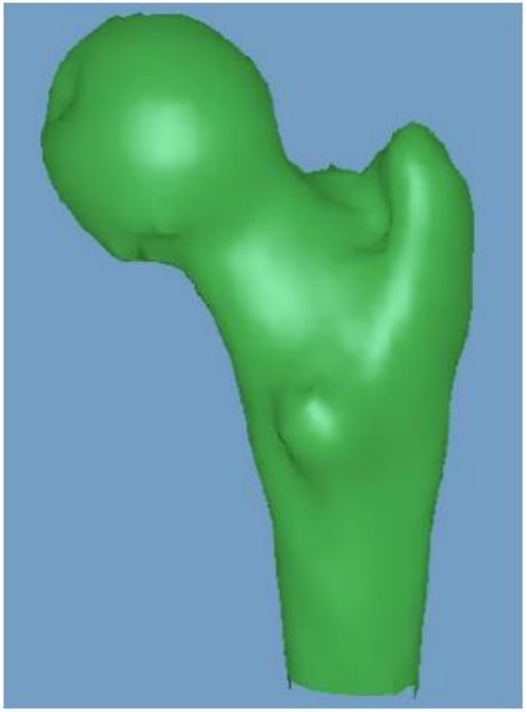
2) MST: $O(n \log n)$

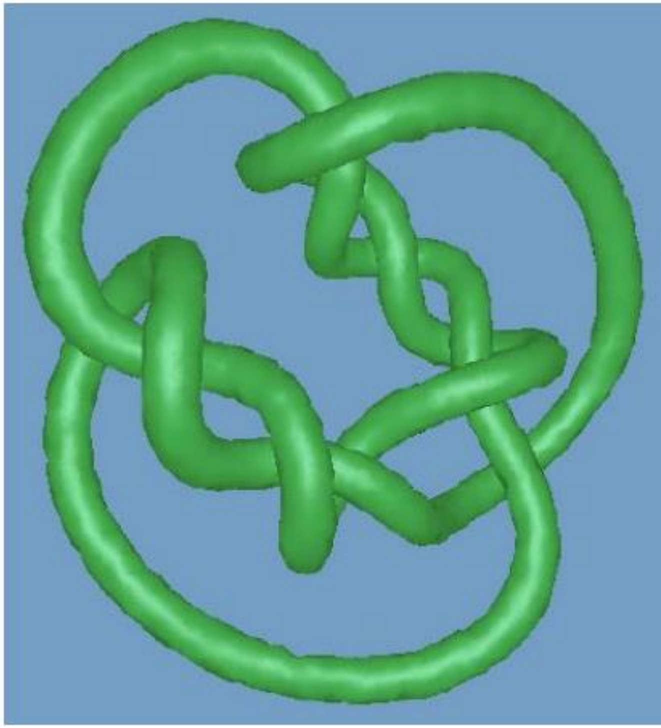
3) Normal Propagation: $O(n)$

4) Distance Function Evaluation: $O(1)$

Results







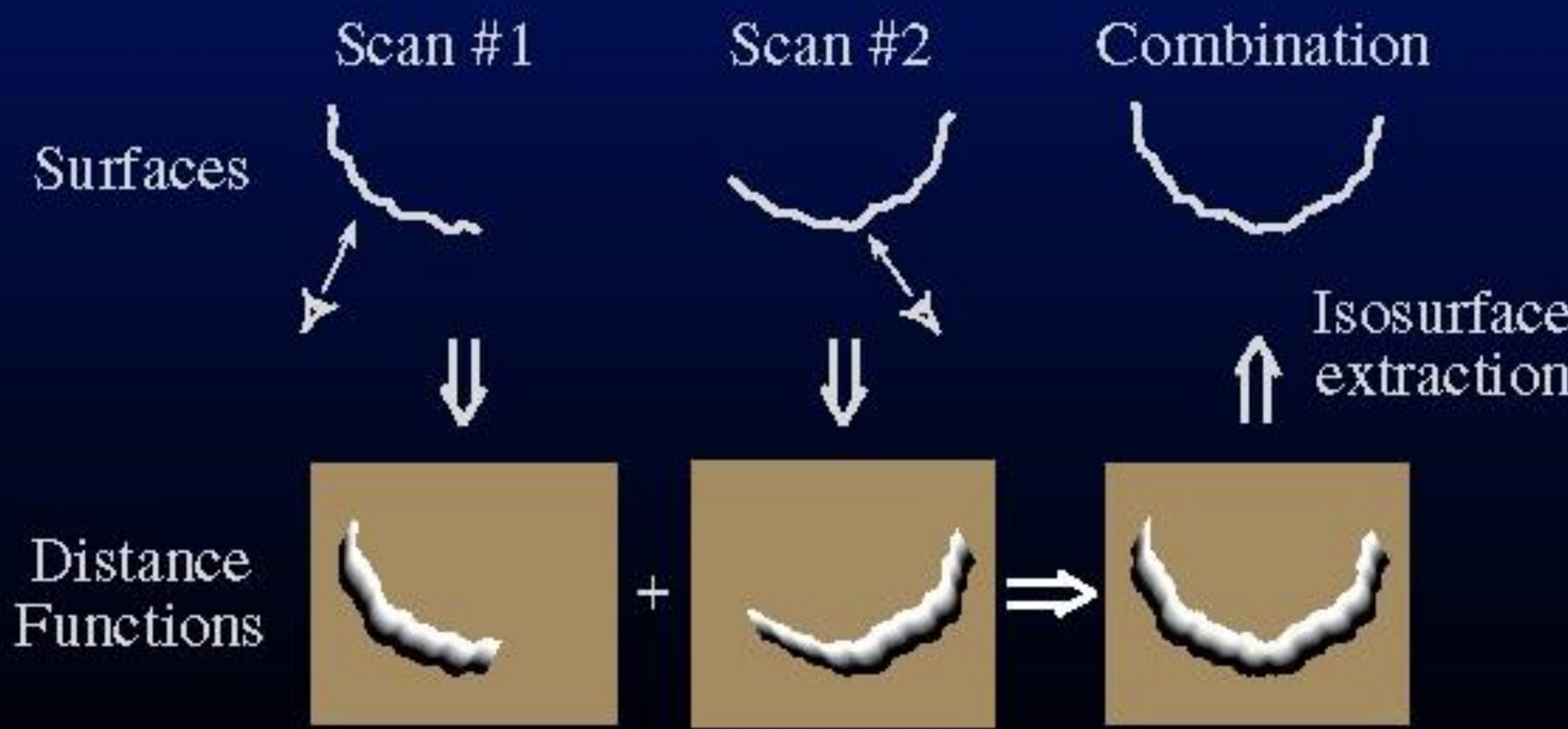
A Volumetric Method for Building Complex Models from Range Images

[Brian Curless and Marc Levoy, SIGGRAPH 1996]

https://graphics.stanford.edu/papers/volrange/slides/

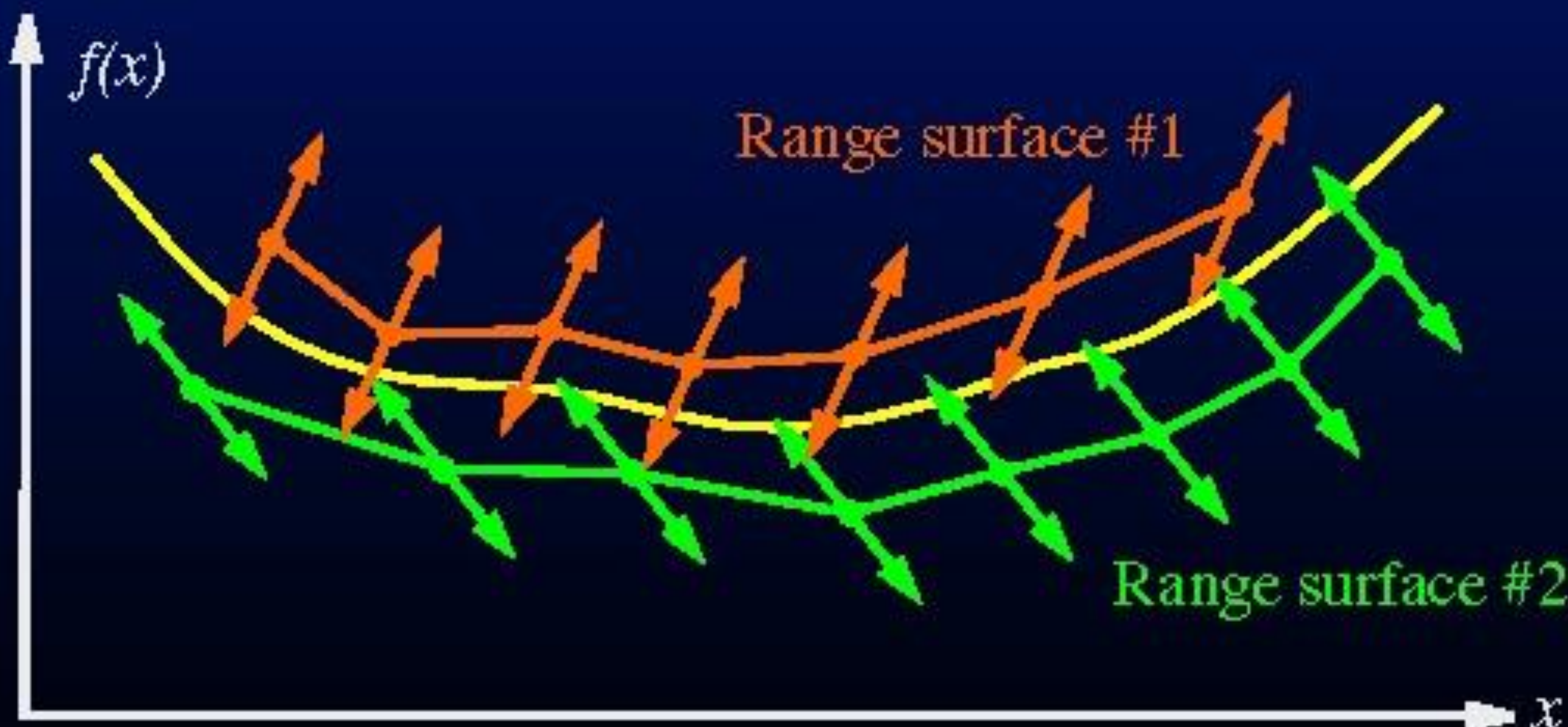
Volumetric method

- For a set of range images, $R_1, R_2, \ldots, R_N$, we construct signed distance functions $d_1(\mathbf{x}), d_2(\mathbf{x}), \ldots, d_N(\mathbf{x})$.
- We combine these functions to generate the cumulative function, $D(\mathbf{x})$.
- We extract the desired manifold as the isosurface, $D(\mathbf{x}) = 0$.



Scan #1
Scan #2
Combination
Surfaces
Isosurface extraction
Distance Functions
+

Least squares solution



$$E(f) = \sum_{i=1}^{N} \underbrace{\int \overbrace{d_i^2(x, f)}^{\text{Error per point}} dx}_{\text{Error per range surface}}$$

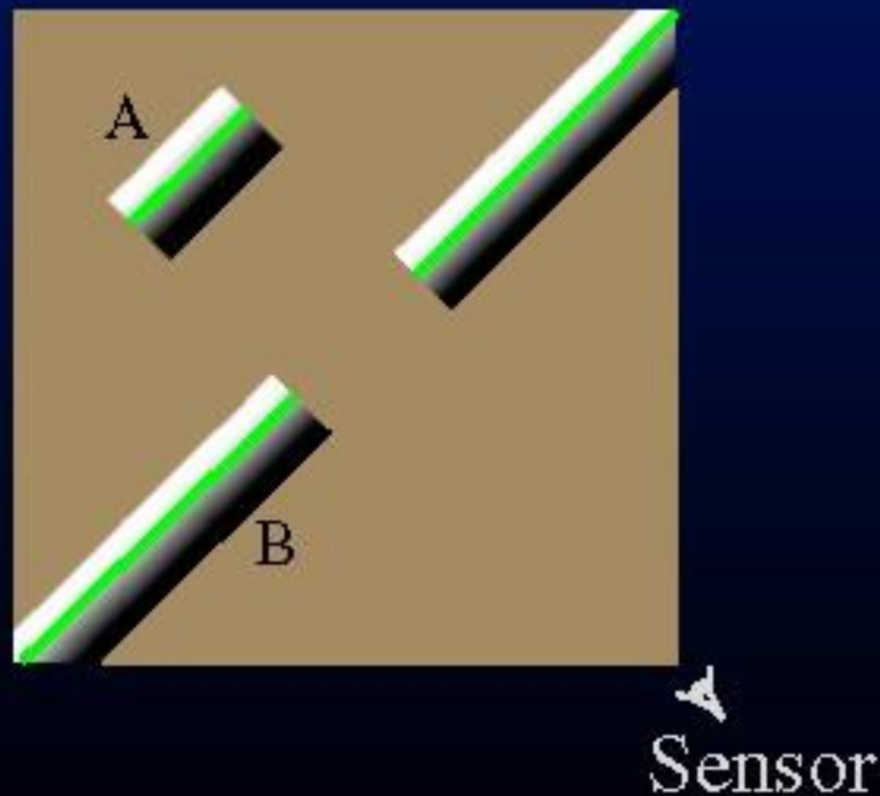
Finding the $f(x)$ that minimizes E yields the optimal surface.

This $f(x)$ is exactly the zero-crossing of the combined signed distance functions.

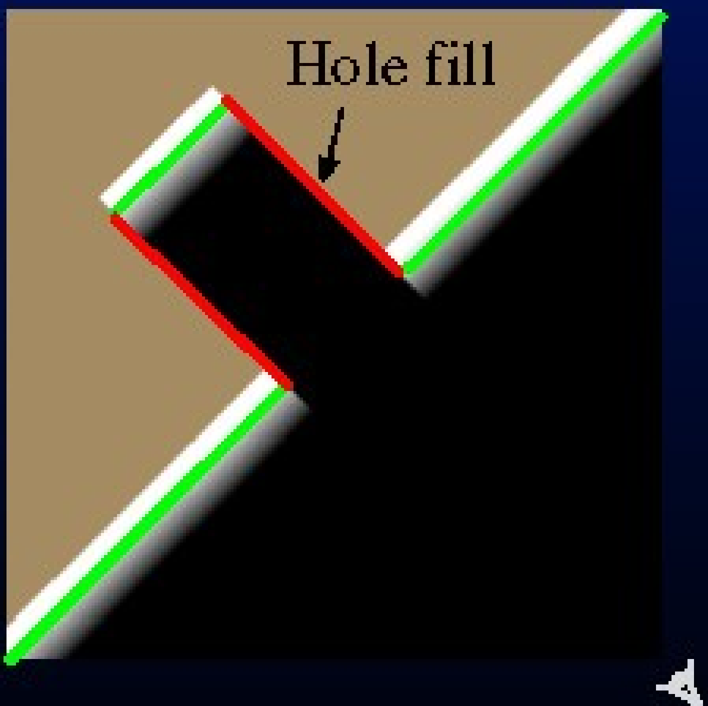
Hole Filling Using Sensor Information

- It is possible to fill holes in the polygonal mesh directly, but such methods:
  - Are hard to make robust
  - Do not use all available information

Without space carving



With space carving

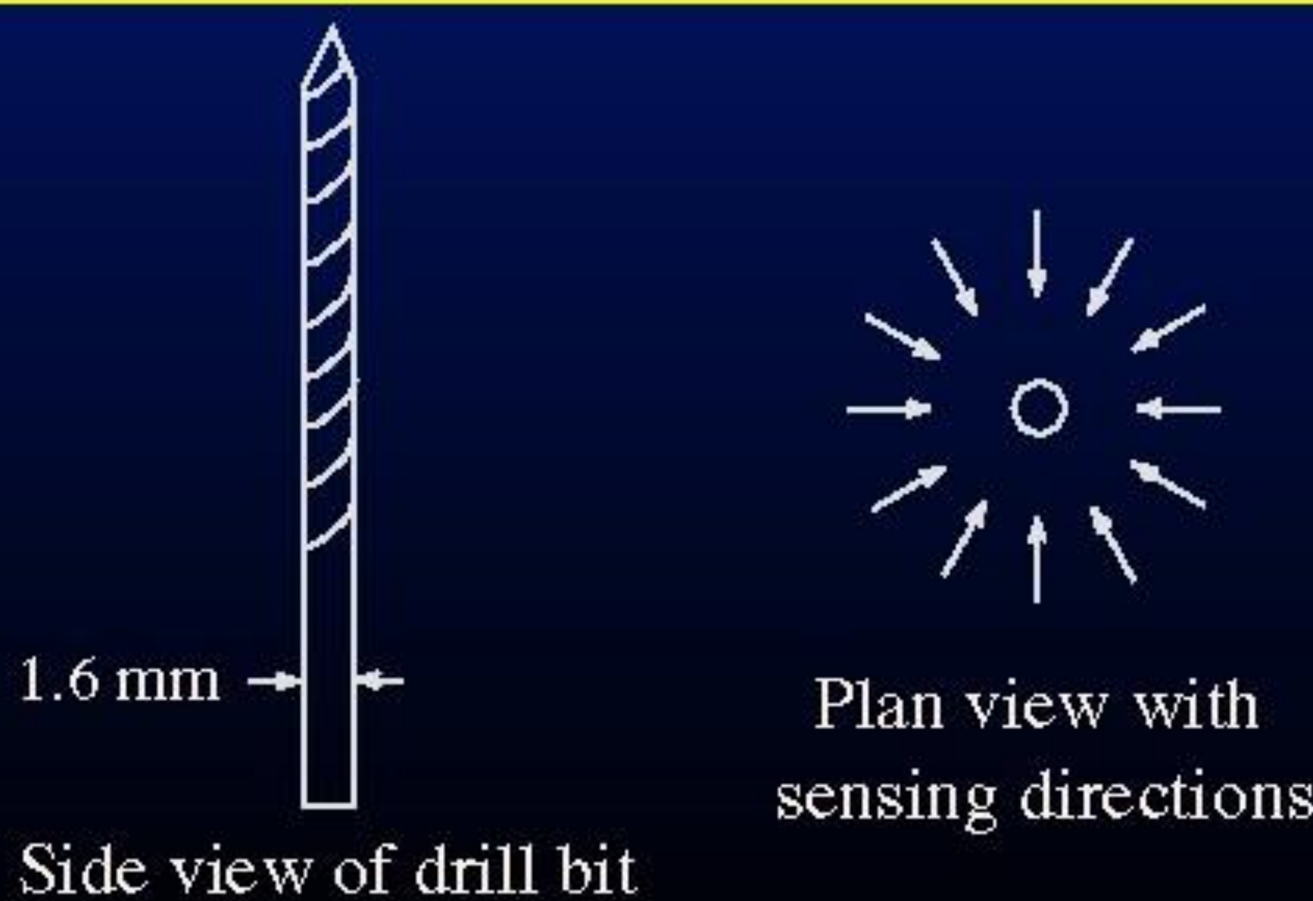


Unseen

Empty

Near surface

Drill bit


1.6 mm
Side view of drill bit
Plan view with
sensing directions

Plan view

Unorganized points

Range surfaces

Zippered mesh

Volumetric mesh

Side view

Photograph of painted drill bit

Zippered mesh

Volumetric mesh

...to hardcopy

Before
hole filling

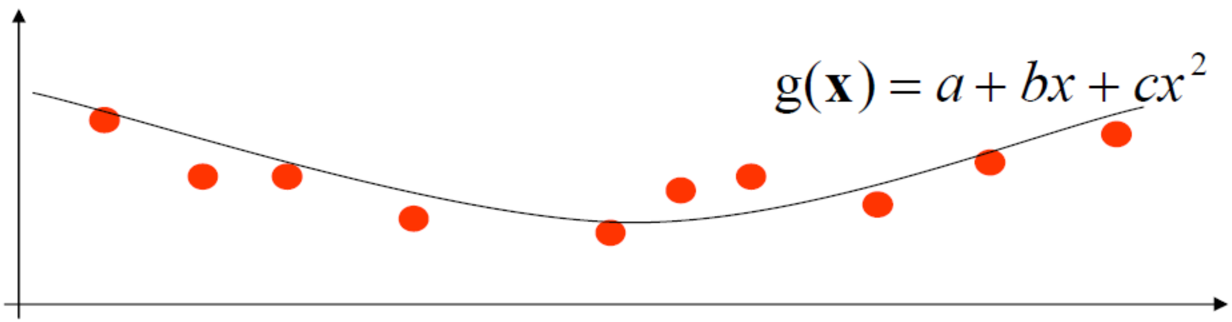
After
hole filling

Hardcopy

Other Print-Based Fitting Methods

Least Squares

- Fit a primitive to the data
- Minimizes squared distances between the points and the primitive



$$\min_{g} \sum_{i} \left(f_i - g(\mathbf{p}_i)\right)^2$$

Least Squares-- Example

- Primitive is a (univariate) polynomial

$$g(x) = (1, x, x^2, \ldots) \cdot \mathbf{c}^T$$

$$\min \sum_i \left(f_i - (1, p_i, p_i^2, \ldots)\mathbf{c}^T\right)^2 \Rightarrow$$

$$0 = \sum_i 2 p_i^j \left(f_i - (1, p_i, p_i^2, \ldots)\mathbf{c}^T\right)$$

- Linear system of equations

Least Squares-- Example

- Resulting system

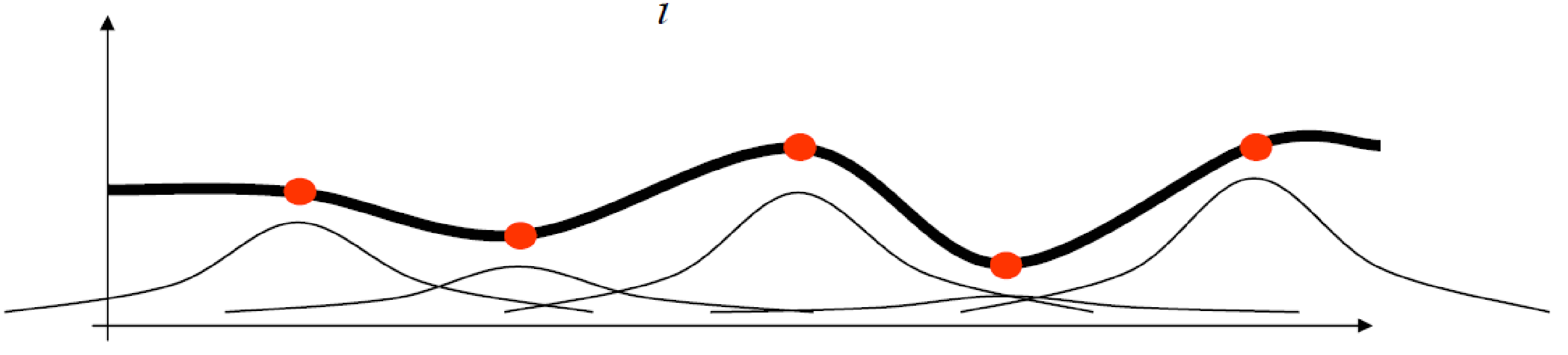
$$0 = \sum_i 2 p_i^j \left(f_i - \left(1, p_i, p_i^2, \ldots\right) \mathbf{c}^T \right) \Leftrightarrow$$

$$\sum_i \begin{pmatrix} 1 & p_i & p_i^2 & \cdots \\ p_i & p_i^2 & p_i^3 & \\ p_i^2 & p_i^3 & p_i^4 & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} = 2 \sum_i f_i \begin{pmatrix} 1 \\ p_i \\ p_i^2 \\ \vdots \end{pmatrix}$$

Radial Basis Functions

- Represent approximating function as
  - Sum of radial functions *r*
  - Centered at the data points p_i

$$f(\mathbf{x}) = \sum_i w_i \, r(\|\mathbf{p}_i - \mathbf{x}\|)$$



Radial Basis Functions

- Solve $$f_j = \sum_i w_i r\left(\left\|\mathbf{p}_i - \mathbf{p}_j\right\|\right)$$
  to compute weights w_i
- Linear system of equations

$$\begin{pmatrix} \mathrm{r}(0) & \mathrm{r}\left(\left\|\mathbf{p}_0 - \mathbf{p}_1\right\|\right) & \mathrm{r}\left(\left\|\mathbf{p}_0 - \mathbf{p}_2\right\|\right) & \cdots \\ \mathrm{r}\left(\left\|\mathbf{p}_1 - \mathbf{p}_0\right\|\right) & \mathrm{r}(0) & \mathrm{r}\left(\left\|\mathbf{p}_1 - \mathbf{p}_2\right\|\right) & \\ \mathrm{r}\left(\left\|\mathbf{p}_2 - \mathbf{p}_0\right\|\right) & \mathrm{r}\left(\left\|\mathbf{p}_2 - \mathbf{p}_1\right\|\right) & \mathrm{r}(0) & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}$$

Radial Basis Functions

- Solvability depends on radial function
- Several choices assure solvability
  - $\mathrm{r}(d) = d^2 \log d$ (thin plate spline)
  - $\mathrm{r}(d) = e^{-d^2/h^2}$ (Gaussian)
    - h is a data parameter
    - h reflects the feature size or anticipated spacing among points

Function Spaces!

- Monomial, Lagrange, RBF share the same principle
  - Choose basis of a function space
  - Find weight vector for base elements by solving linear system defined by data points
  - Compute values as linear combinations

- Properties
  - One costly preprocessing step
  - Simple evaluation of function in any point

Function Spaces?

- Problems
  - Many points lead to larger linear systems
  - Evaluation requires global solutions
- Solutions
  - RBF with compact support
    - Matrix is sparse
    - Still: solution depends on every data point, though drop-off is exponential with distance
  - Local approximation approaches

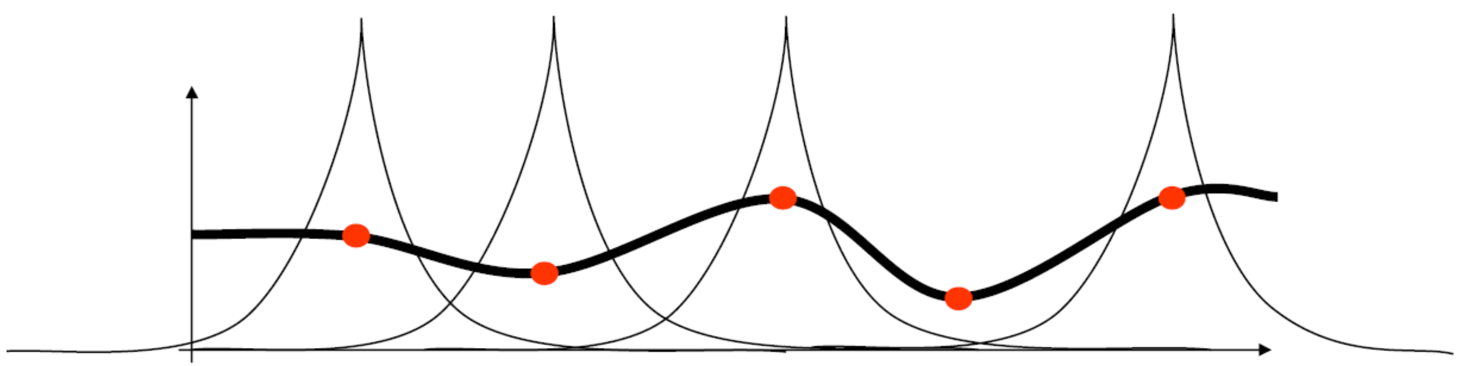
Partition of Unity

Shepard Interpolation

- Approach $\mathrm{f}(\mathbf{x}) = \sum_i \phi_i(\mathbf{x}) f_i$

  with basis functions $\phi_i(\mathbf{x}) = \dfrac{\left\|\mathbf{x}-\mathbf{x}_i\right\|^{-p}}{\sum_j \left\|\mathbf{x}-\mathbf{x}_j\right\|^{-p}}$

- Define $\mathrm{f}(\mathbf{p}_i) = f_i = \lim_{\mathbf{x}\to\mathbf{p}_i} \mathrm{f}(\mathbf{x})$



Shepard Interpolation

- $\mathrm{f}(\mathbf{x})$ is a convex combination of ϕ_i, because all $\phi_i \in [0,1]$ and $\sum \phi_i(\mathbf{x}) \equiv 1$
- $\mathrm{f}(\mathbf{x})$ is contained in the convex hull of data points
- $\left|\{\mathbf{p}_i\}\right| > 1 \Rightarrow \mathrm{f}(\mathbf{x}) \in \mathrm{C}^\infty \quad \text{and} \quad \nabla \mathrm{f}(\mathbf{p}_i) = \mathbf{0}$

  ➔ Data points are saddles
- global interpolation

  ➔ every $\mathrm{f}(\mathbf{x})$ depends on all data points
- Only constant precision, i.e. only constant functions are reproduced exactly

Shepard Interpolation

Localization:

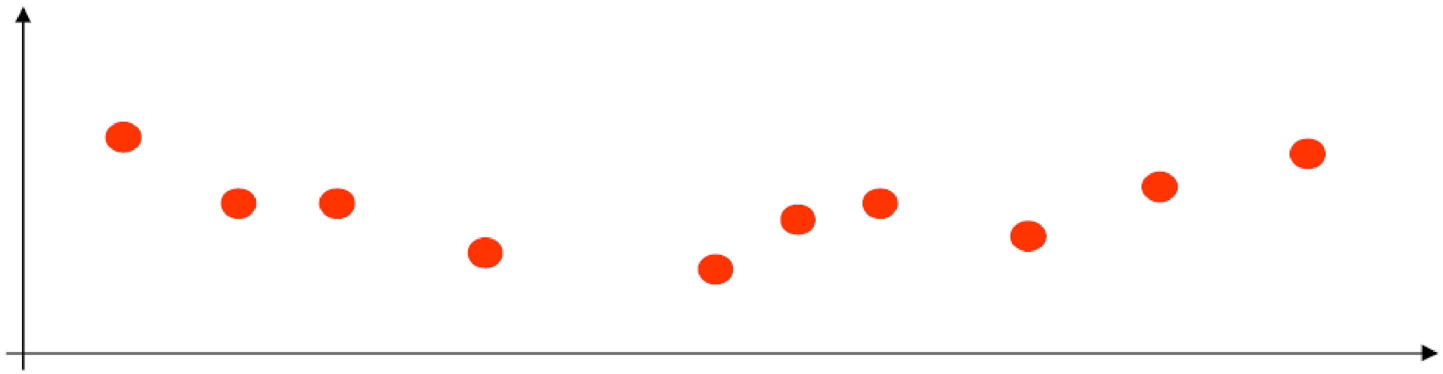
- Set $$\mathrm{f}(\mathbf{x}) = \sum_i \mu_i(\mathbf{x})\phi_i(\mathbf{x}) f_i$$
- with $$\mu_i(\mathbf{x}) = \begin{cases} \left(1 - \|\mathbf{x} - \mathbf{p}_i\| / R_i\right)^{\nu} & \text{if } \|\mathbf{x} - \mathbf{p}_i\| < R_i \\ 0 & \text{else} \end{cases}$$

  for reasonable R_i and $\nu > 1$

  → no constant precision because of possible holes in the data

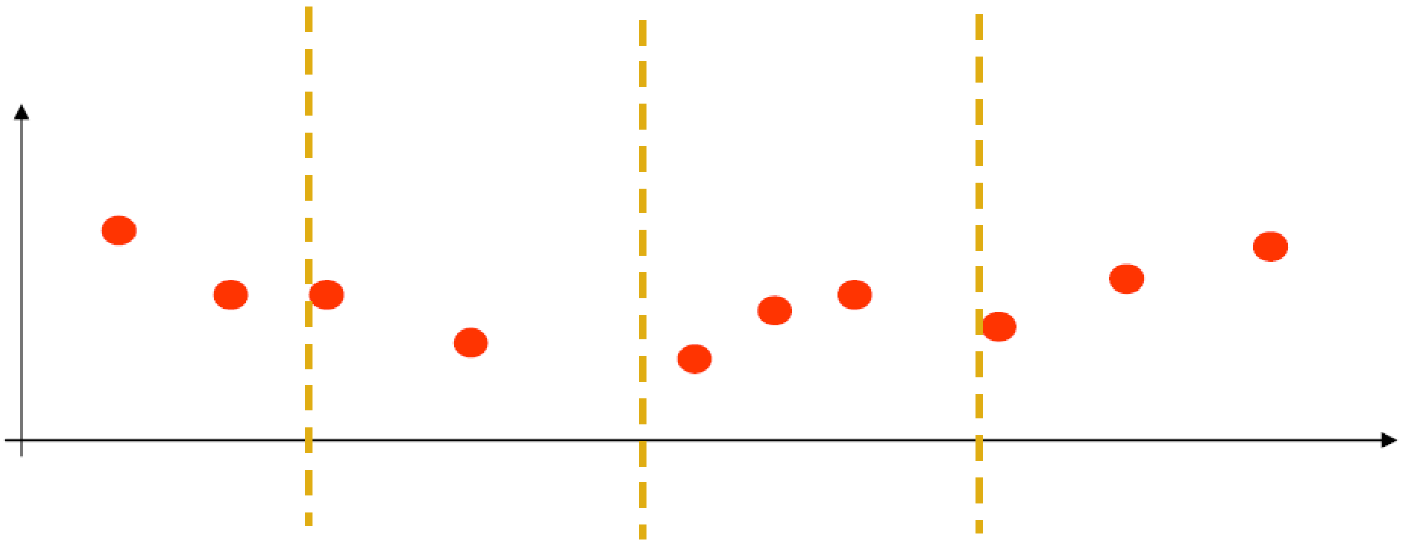
Partition of Unity Methods

Partial of Unity Methods



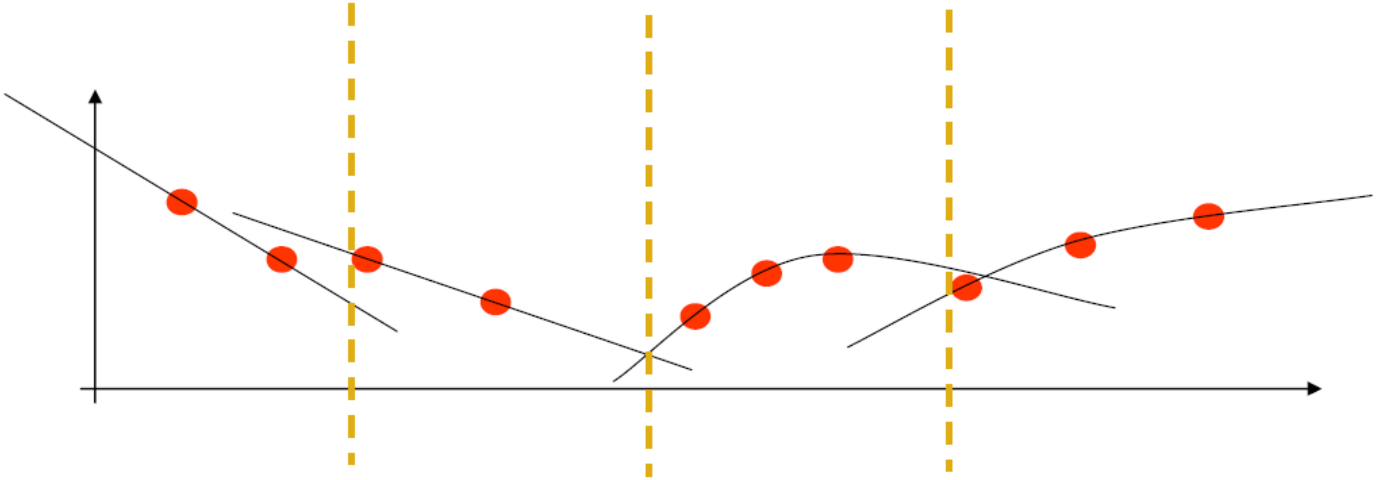
Partial of Unity Methods

Subdivide domain into cells



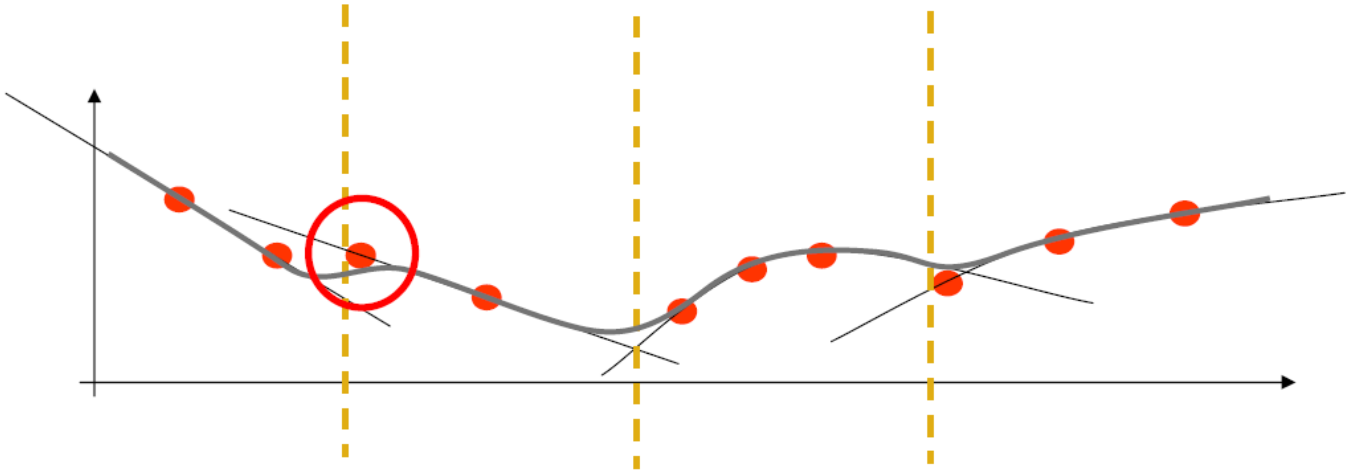
Partial of Unity Methods

Compute local interpolation per cell



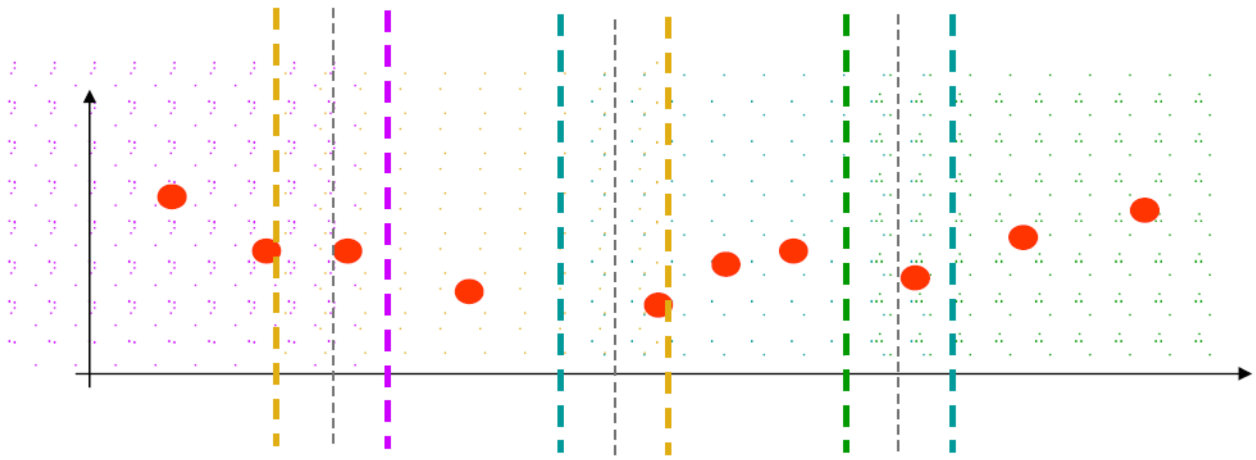
Partial of Unity Methods

Blend local interpolations?



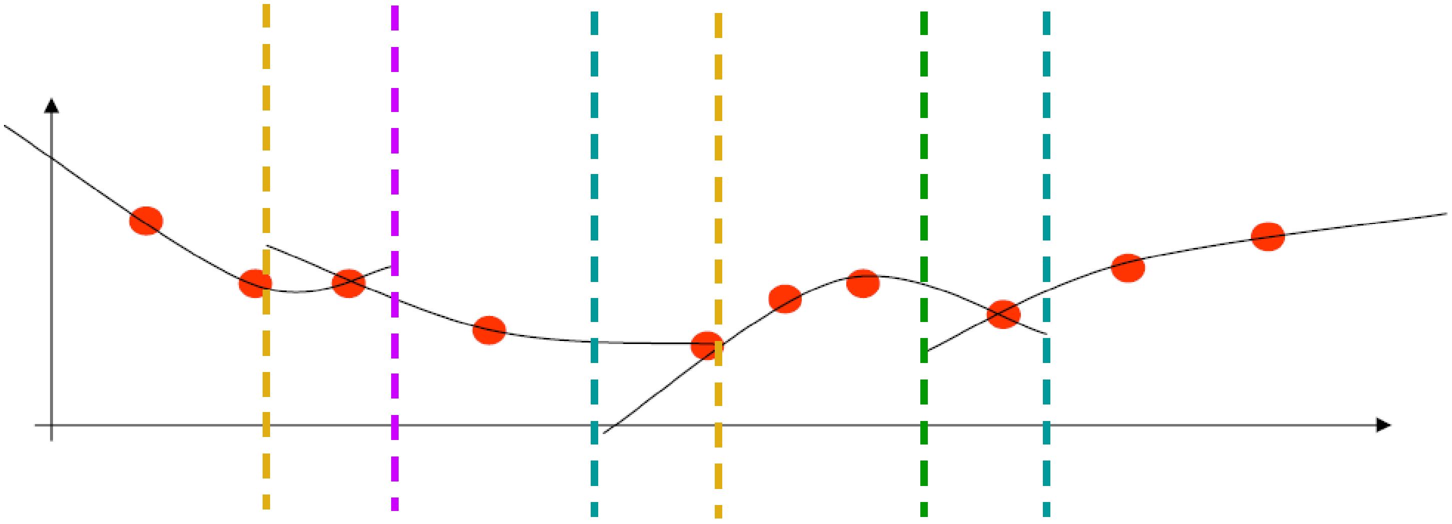
Partial of Unity Methods

Subdivide domain into overlapping cells



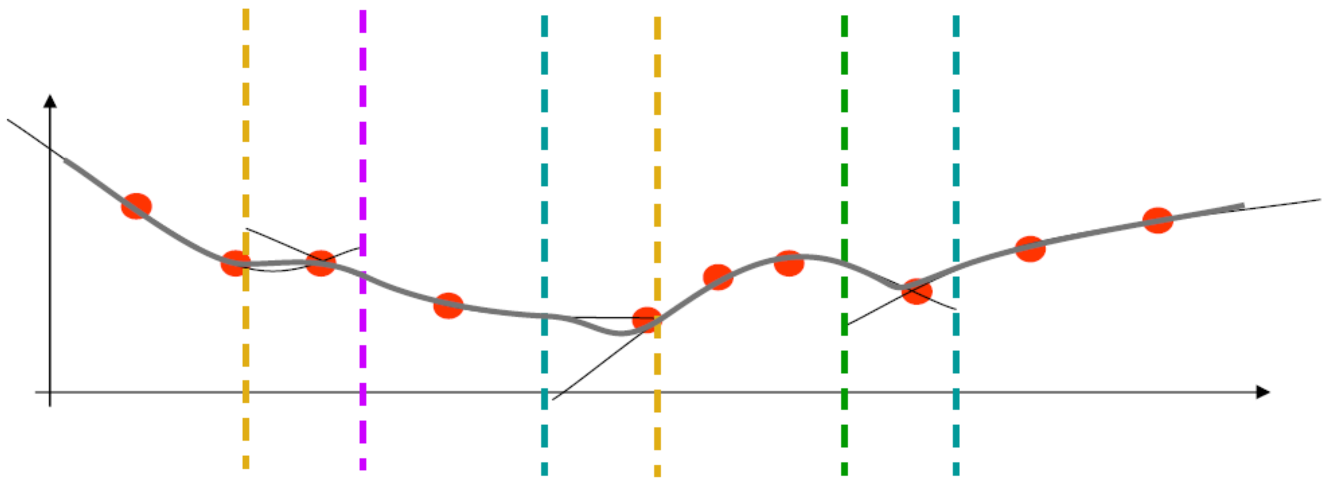
Partial of Unity Methods

Compute local interpolations



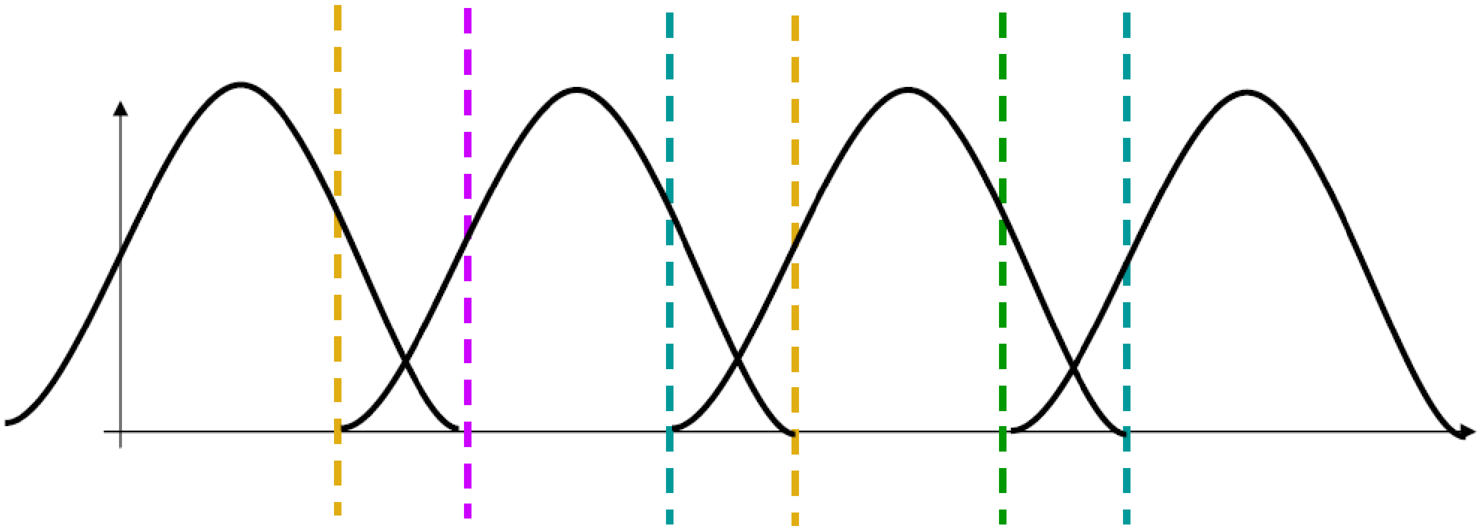
Partial of Unity Methods

Blend local interpolations



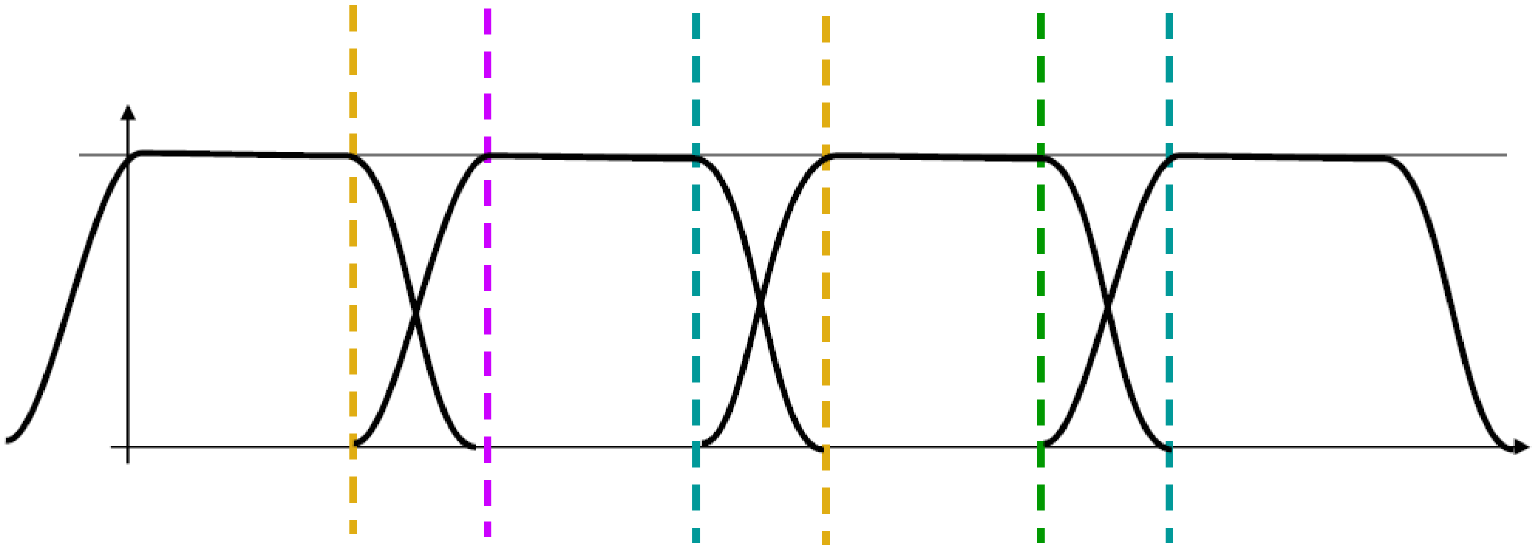
Partition of Unity Methods

- Weights should
  - Have the (local) support of the cell



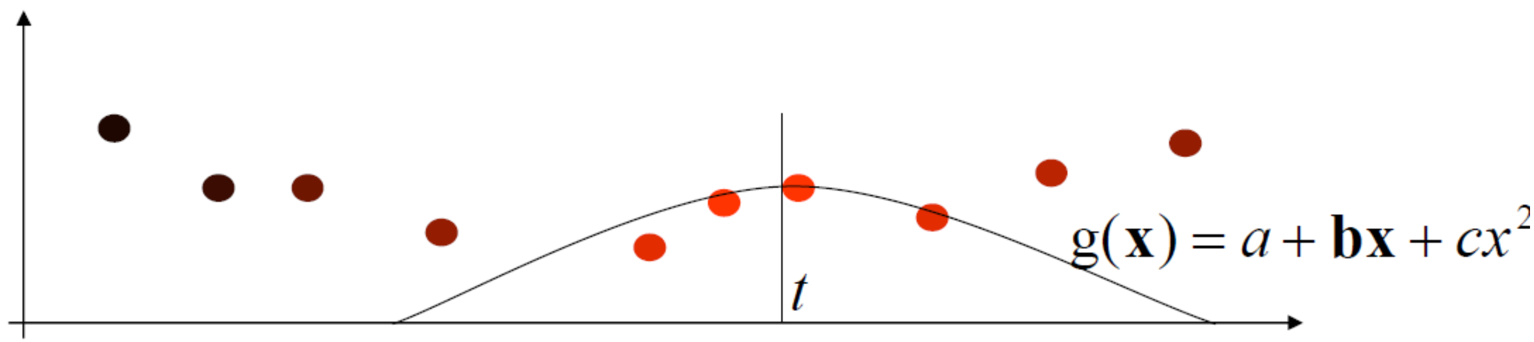
Partition of Unity Methods

- Weights should
  – Sum up to one everywhere (Shepard weights)
  – Have the (local) support of the cell



Moving Least Squares

- Compute a local LS approximation at **x**
- Weight data points based on distance to **x**



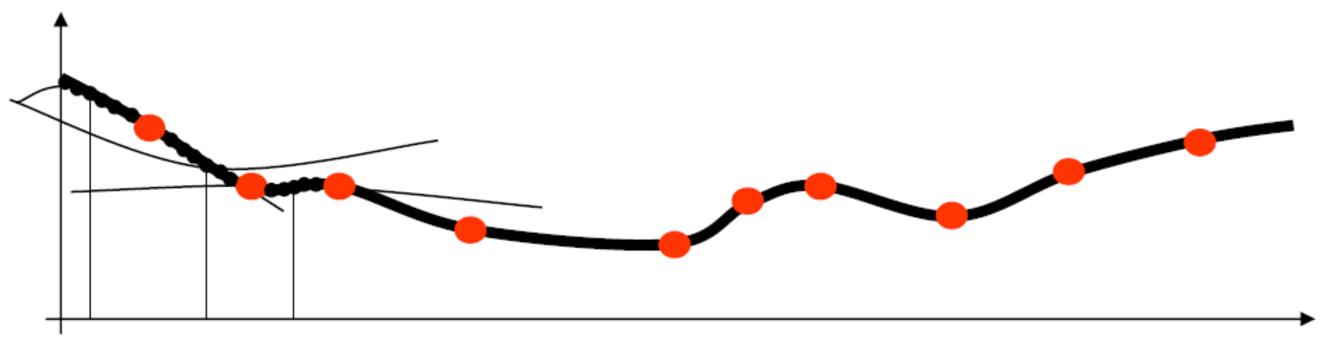
$$\min \sum_i \left(f_i - g(\mathbf{p}_i)\right)^2 \theta\left(\left\|\mathbf{x} - \mathbf{p}_i\right\|\right)$$

Moving Least Squares

- The set

$$f(\mathbf{x}) = g_{\mathbf{x}}(\mathbf{x}), g_{\mathbf{x}} : \min_{g} \sum_{i} \left(f_i - g(\mathbf{p}_i) \right)^2 \theta(\|\mathbf{x} - \mathbf{p}_i\|)$$

  is a smooth curve, iff θ is smooth



Moving Least Squares

- Typical choices for θ:
  - $\theta(d) = d^{-r}$
  - $\theta(d) = e^{-d^2/h^2}$

- Note: $\theta_i = \theta(\|\mathbf{x} - \mathbf{p}_i\|)$ is fixed
- For each $\mathbf{x}$
  - Standard weighted LS problem
  - Linear iff corresponding LS is linear

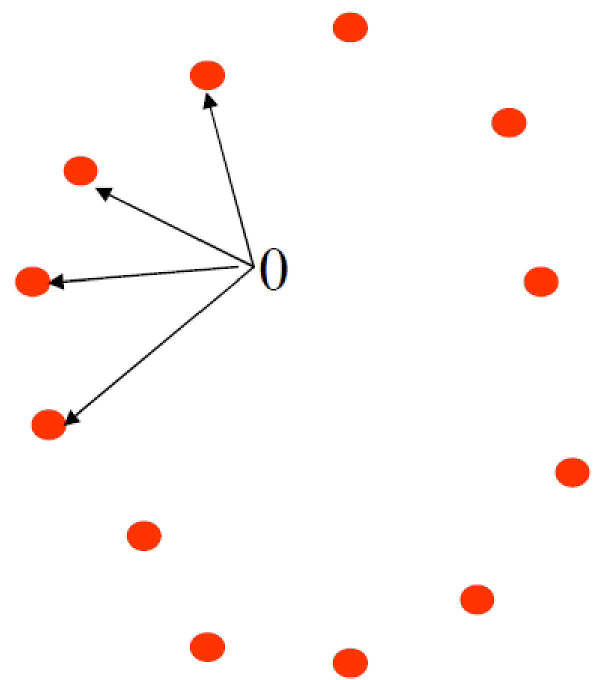
Fitting

Implicits

- Each orientable 2-manifold can be embedded in 3-space
- Idea: Represent 2-manifold as zero-set of a scalar function in 3-space
  - Inside: $f(\mathbf{x}) < 0$
  - On the manifold: $f(\mathbf{x}) = 0$
  - Outside: $f(\mathbf{x}) > 0$

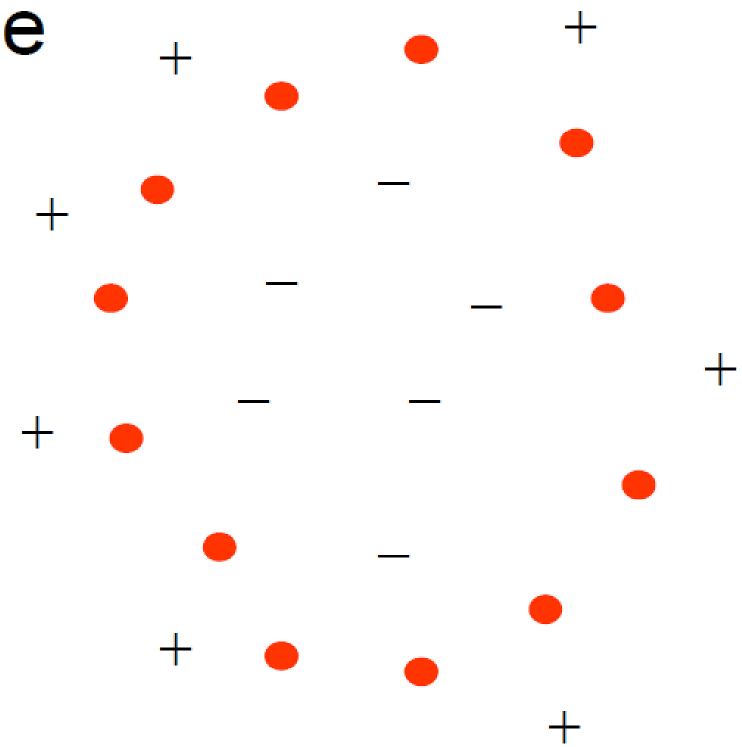
Implicits from point samples

- Function should be zero in data points
  - $f(\mathbf{p}_i) = 0$
- Use standard approximation techniques to find f
- Trivial solution: $f = 0$
- Additional constraints are needed



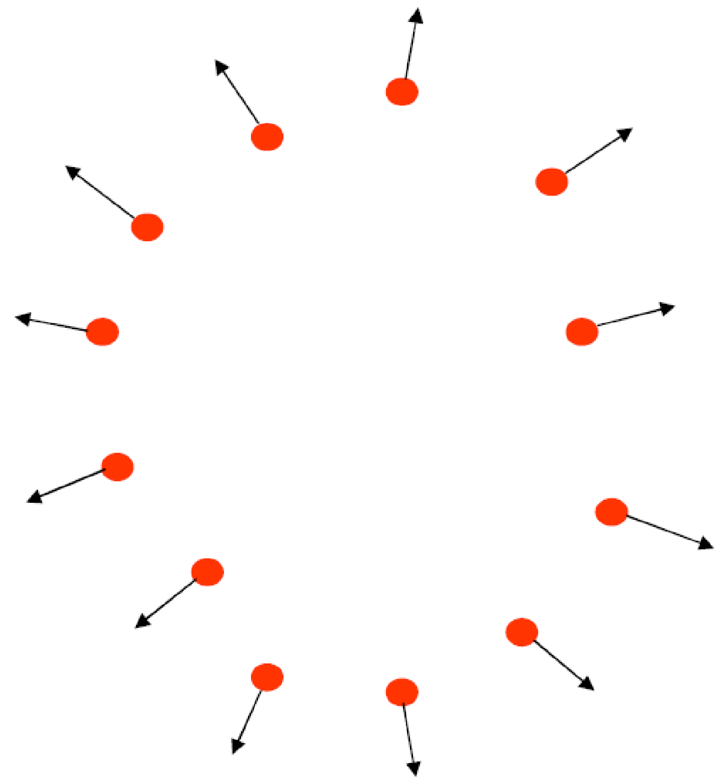
Implicits from point samples

- Constraints define inside and outside
- Simple approach (Turk, O'Brien)
  - Sprinkle additional information manually
  - Make additional information soft constraints



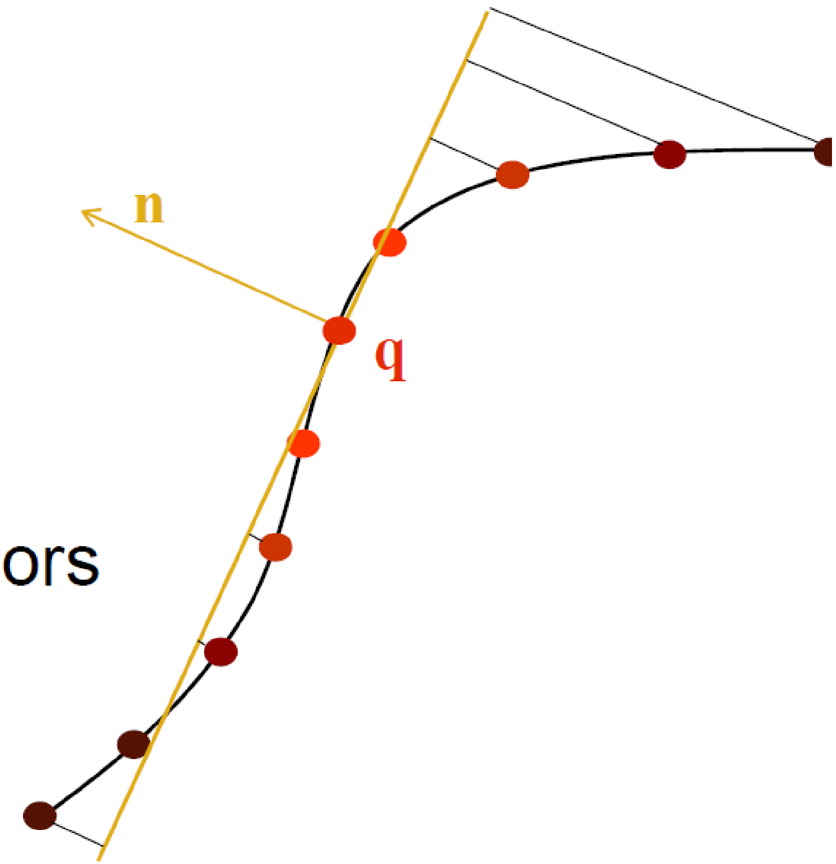
Implicits from point samples

- Use normal information
- Normals could be computed from scan
- Or, normals have to be estimated



Estimating normals

- Normal orientation (Implicits are signed)
  - Use inside/outside information from scan
- Normal direction by fitting a tangent
  - LS fit to nearest neighbors
  - Weighted LS fit
  - MLS fit

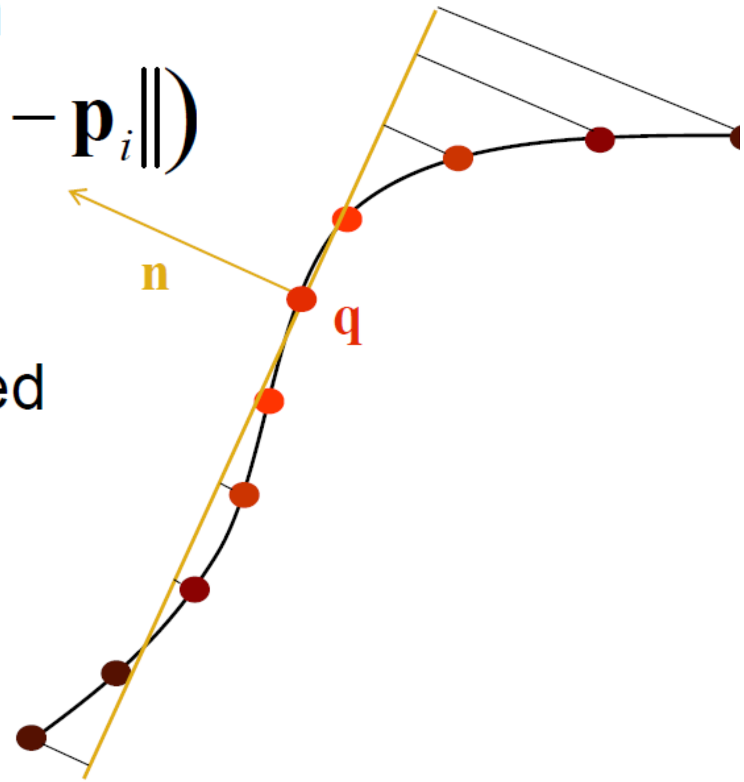


Estimating normals

- General fitting problem

$$\min_{\|\mathbf{n}\|=1} \sum_i \langle \mathbf{q} - \mathbf{p}_i, \mathbf{n} \rangle^2 \theta(\|\mathbf{q} - \mathbf{p}_i\|)$$

  – Problem is non-linear because $\mathbf{n}$ is constrained to unit sphere



Estimating normals

- The constrained minimization problem

$$\min_{\|\mathbf{n}\|=1} \sum_i \langle \mathbf{q} - \mathbf{p}_i, \mathbf{n} \rangle^2 \theta_i$$

is solved by the eigenvector corresponding to the smallest eigenvalue of the following co-variance matrix

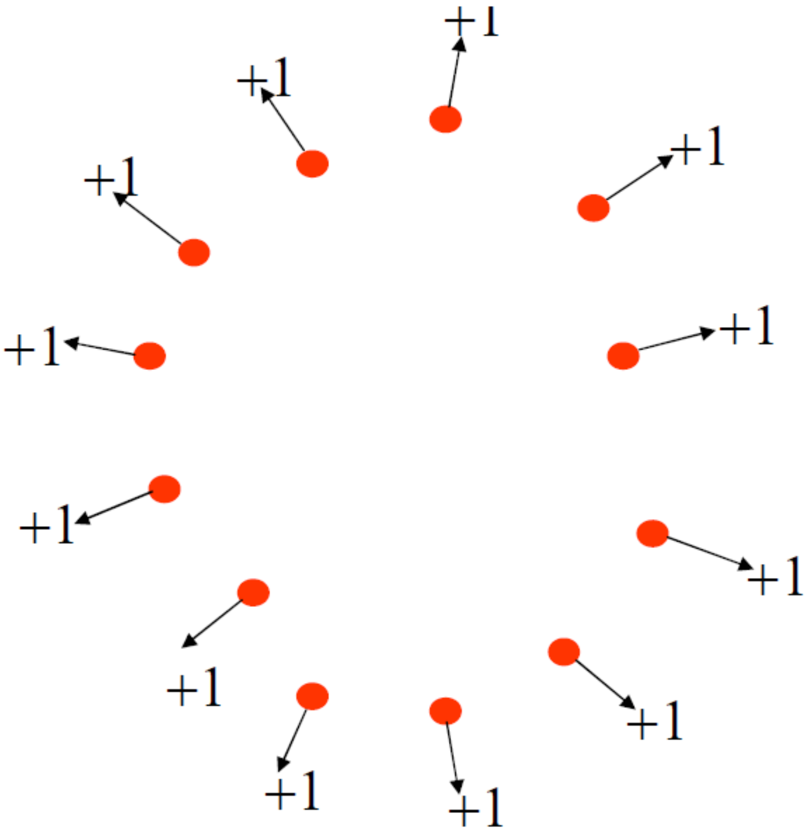
$$\sum_i (\mathbf{q} - \mathbf{p}_i) \cdot (\mathbf{q} - \mathbf{p}_i)^{\mathrm{T}} \theta_i$$

which is constructed as a sum of weighted outer products.

Implicits from point samples

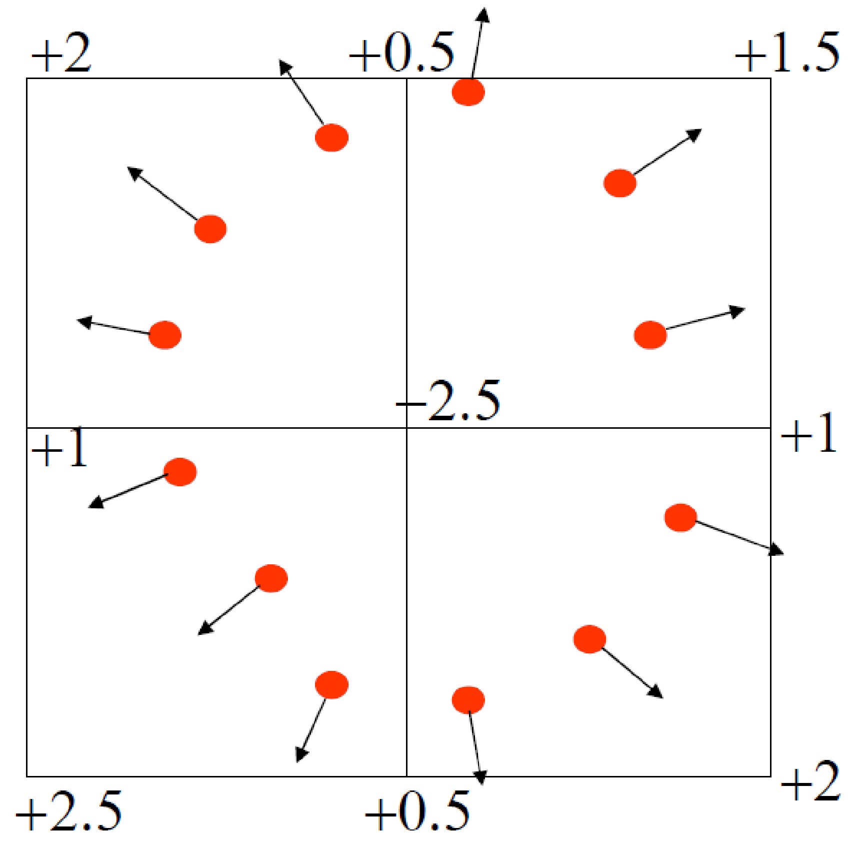
- Compute non-zero anchors in the distance field
- Use normal information directly as constraints

$$f(\mathbf{p}_i + \mathbf{n}_i) = 1$$



Implicits from point samples

- Compute non-zero anchors in the distance field
- Compute distances at specific points
  - Vertices, mid-points, etc. in a spatial subdivision



Computing Implicits

- Given N points and normals $\mathbf{p}_i, \mathbf{n}_i$ and constraints $\mathrm{f}(\mathbf{p}_i) = 0, \mathrm{f}(\mathbf{c}_i) = d_i$

- Let $\mathbf{p}_{i+N} = \mathbf{c}_i$
- An RBF approximation

$$\mathrm{f}(\mathbf{x}) = \sum_i w_i \theta(\|\mathbf{p}_i - \mathbf{x}\|)$$

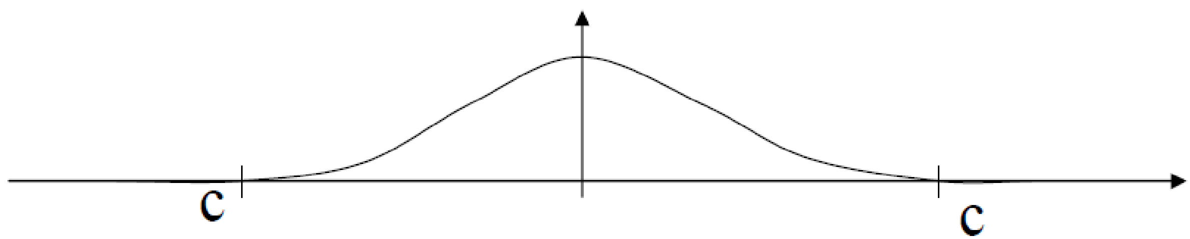
leads to a system of linear equations

Computing Implicits

- Practical problems: $N > 10000$
- Matrix solution becomes difficult
- Two solutions
  - Sparse matrices allow iterative solution
  - Smaller number of RBFs

Computing Implicits

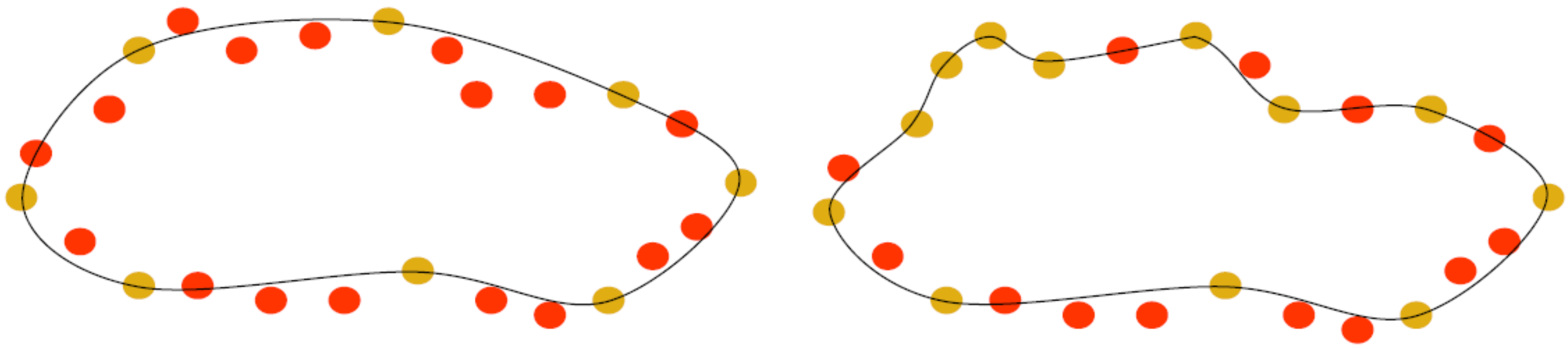
- Sparse matrices
$$\begin{pmatrix} \theta(0) & \theta(\|\mathbf{p}_0 - \mathbf{p}_1\|) & \theta(\|\mathbf{p}_0 - \mathbf{p}_2\|) & \cdots \\ \theta(\|\mathbf{p}_1 - \mathbf{p}_0\|) & \theta(0) & \theta(\|\mathbf{p}_1 - \mathbf{p}_2\|) & \\ \theta(\|\mathbf{p} - \mathbf{p}_0\|) & \theta(\|\mathbf{p}_2 - \mathbf{p}_1\|) & \theta(0) & \\ \vdots & & & \ddots \end{pmatrix}$$
  - Needed: $d > c \rightarrow r(d) = 0, r'(c) = 0$



  - Compactly supported RBFs

Computing Implicits

- Smaller number of RBFs
- Greedy approach (Carr et al.)
  - Start with random small subset
  - Add RBFs where approximation quality is not sufficient



RBF Implicits - Results

Image courtesy: Greg Turk

Multi-Level Partition of Unity

Overview

- Goal:
  - Use multi-level partition of unity (MPU) implicit surface to construct surface models
- 3 Key Concepts:
  - Piecewise quadratic functions used as local estimates
  - Weighing functions that blend these local shape functions.
  - Octree subdivision that adapts based on shape complexity.
- Flexibility
  - Accurate representation of sharp features (edges, corners)
- Adaptive approximation based on required accuracy
  - Determines space/time complexity

Advantages of Implicit Functions

- Edit surfaces using standard implicit modeling operations: shape blending, offsets, deformations

Method Summary: Setup

- Given: set of points with normals to indicate surface orientation

- Partition of unity: set of weighing functions that sum to one at all points in the domain

- MPU implicit: adaptive error-controlled approximation of signed distance function from surface
  - Surface is zero-level of the distance function.

Method Summary: Algorithm

- To create implicit representation:
  - Octree-based subdivision of bounding box for entire point set
  - At each cell, fit a piecewise quadratic function (local shape function)
    - Signed distance function: 0 near points, positive inside, negative outside
  - If shape function isn't accurate enough, subdivide further until desired accuracy is achieved
  - In common boundary between cells, shape functions are blended together according to weights from partition of unity functions

- Global implicit of function is given by blending of local shape functions at the leaves of the octree

Partition of Unity

- Generate weight functions:
  - For approximation: use quadratic B-spline b(t)

$$w_i(x) = b\left(\frac{3\left|x - c_i\right|}{2R_i}\right)$$

  - For interpolation: use inverse-distance singular weights

$$w_i(x) = \left[\frac{(R_i - |x - c_i|)_+}{R_i|x - c_i|}\right]^2, \text{ where } (a)_+ = \begin{cases} a & a > 0 \\ 0 & else \end{cases}$$

- For interpolation: use inverse-distance singular weights

$$\Omega \subset \cup_i \; \text{supp}(w_i)$$

Partition of Unity

- Blend local functions using smooth, local weights that add up to 1
  - Partition of unity functions

$$\sum_i \phi_i \equiv 1 \text{ on } \Omega \qquad \phi_i(x) = \frac{w_i(x)}{\sum_{j=1}^{n} w_j(x)}$$

- Define set of local shape functions V_i
- Approximation of a function defined on domain

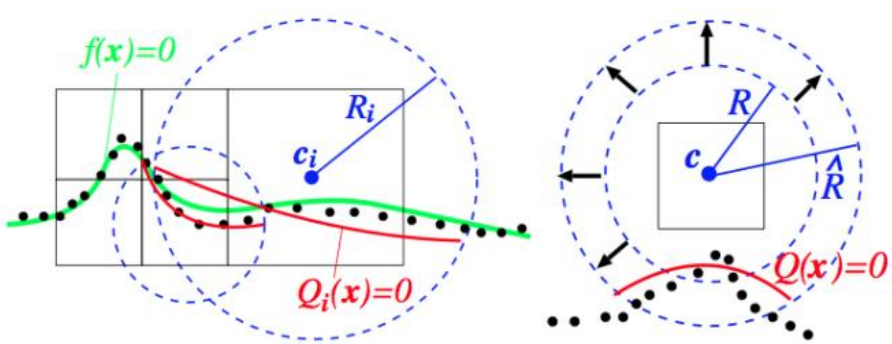
$$Q_i \in V_i \qquad f(x) \approx \sum_i \phi_i(x) Q_i(x)$$

Adaptive Octree

- Octree-based subdivision. Each cell has center c and diagonal length d
- Define the support radius for the cell's weight function: $R = \alpha d$
  - Bigger alpha -> smoother interpolation/approximation, slower computation
  - Time complexity quadratic on alpha
- Must have at least N_{min} points in the sphere to estimate shape function
  - If not enough, iteratively increase radius

$$\hat{R} = \hat{R} + \lambda R$$

$\alpha = 0.75 \qquad \lambda = 0.1$

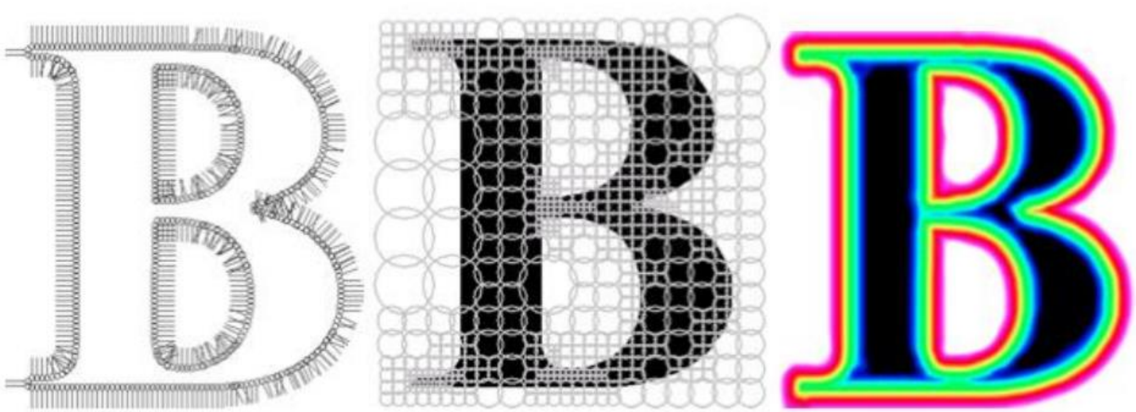


Adaptive Octree

- Local max-norm approximation error estimated based on Taubin distance

$$\epsilon = \max_{|p_i - c| < R} \frac{|Q(p_i)|}{|\nabla Q(p_i)|}$$

- If error is larger than a threshold ε0 , subdivide the cell



Algorithm: Pseudocode

EvaluateMPUapprox$(\mathbf{x}, \varepsilon_0)$
  $S_{wQ} = S_w = 0;$
  root->MPUapprox$(\mathbf{x}, \varepsilon_0)$;
  return S_{wQ}/S_w;

$$S_{wQ} = \sum w_i(x) Q_i(x)$$

$$S_w = \sum w_i(x)$$

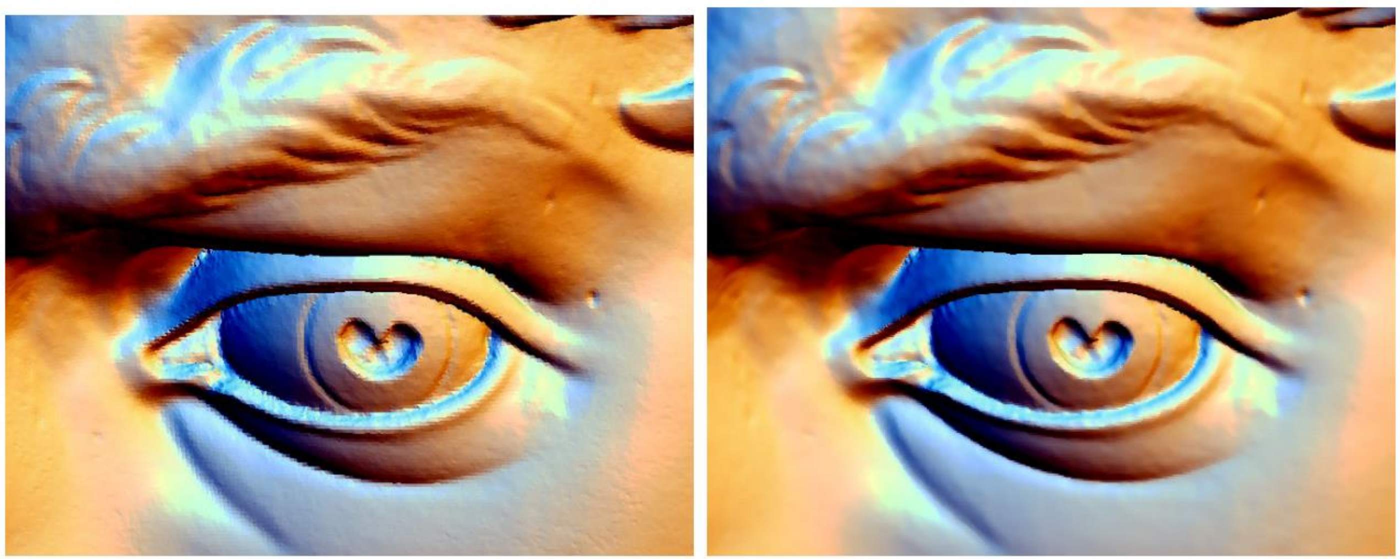
$$f(x) \approx \frac{S_{wQ}}{S_w} = \frac{\sum w_i(x) Q_i(x)}{\sum w_i(x)}$$

MPUapprox$(\mathbf{x}, \varepsilon_0)$
  $d = \|\mathbf{c} - \mathbf{x}\|;$
  if $(d > R)$ then
    return;
  end if
  if (Q is not created yet) then
    Create Q and compute ε;
  end if
  if $(\varepsilon > \varepsilon_0)$ then
    if (No childs) then
      Create childs;
    end if
    for each child
      child->MPUapprox$(\mathbf{x}, \varepsilon_0)$;
    end for
  else
    $S_{wQ} = S_{wQ} + w(d, R) * Q(\mathbf{x});$
    $S_w = S_w + w(d, R);$
  end if

Local Shape Functions – Details in the paper

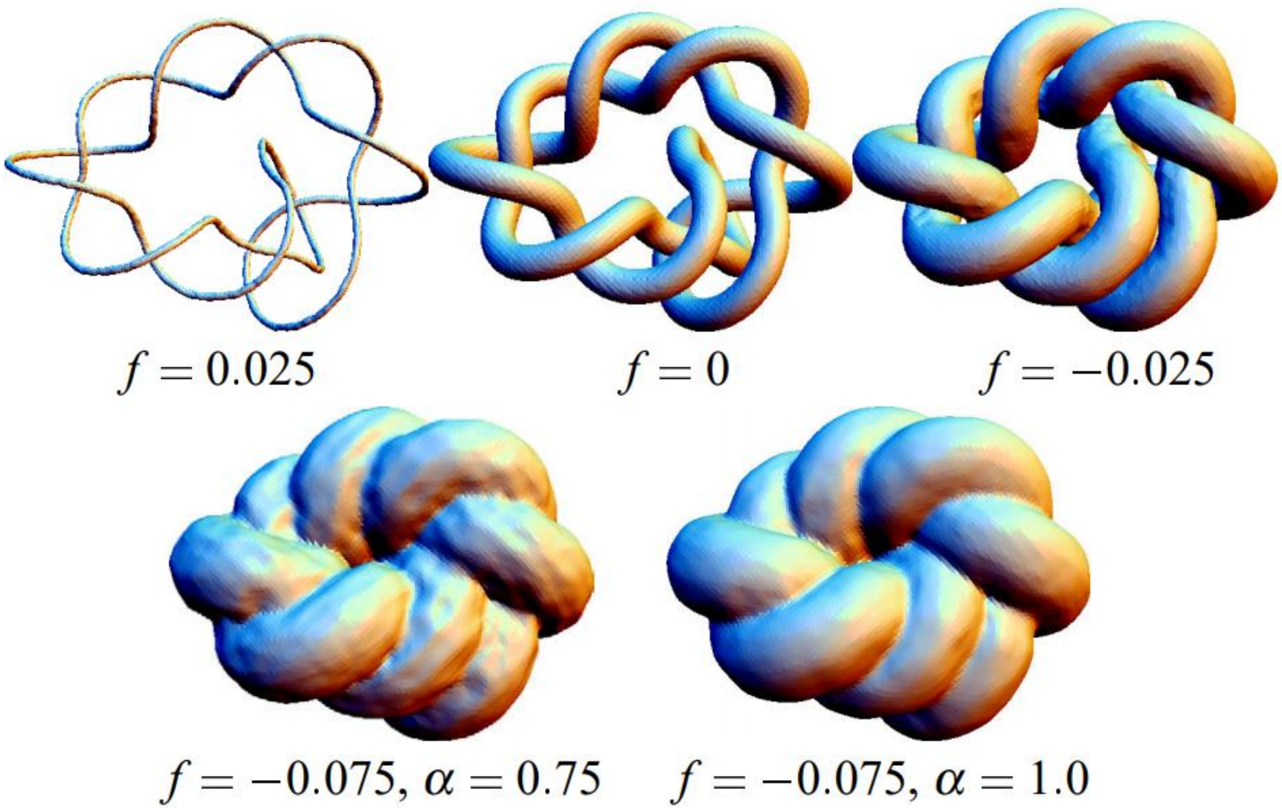
- General 3D quadric
  - Larger parts of the surface: unbounded, more than one sheet B
- Bivariate quadratic polynomial in local coordinates
  - Local smooth patch C
- Piecewise quadric surface to fit sharp features
  - Edges, corners

Representation power



Eye from Stanford's reconstruction of Michalangelo's David (scanned at 1mm resolution). Right: The eye is reconstructed as an MPU implicit with relative accuracy 10^{-4}

Robustness

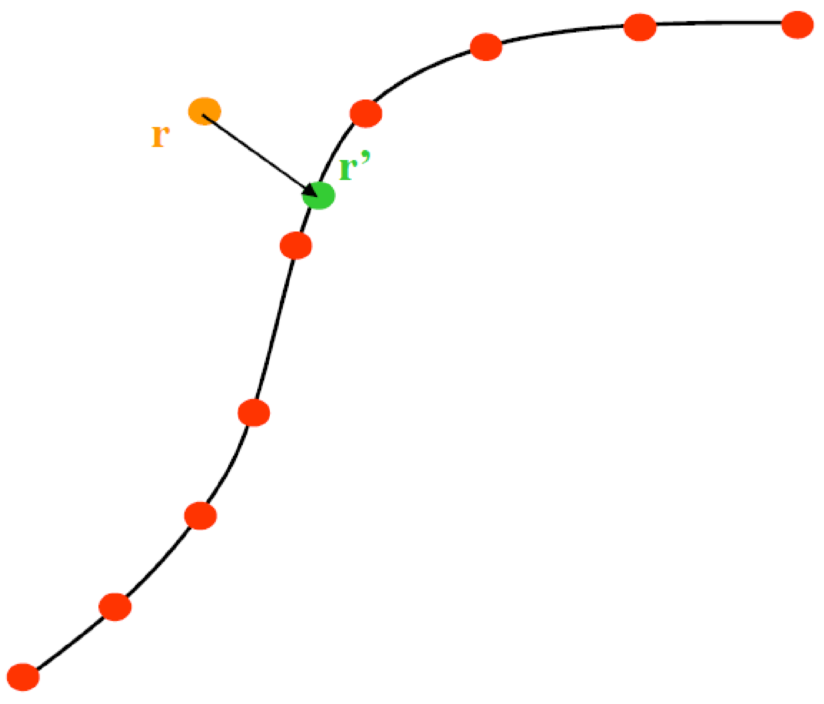


Offsetting of a knot model. The distance function to the knot is approximated by w=f(x,y,z)

Projection-based Approaches

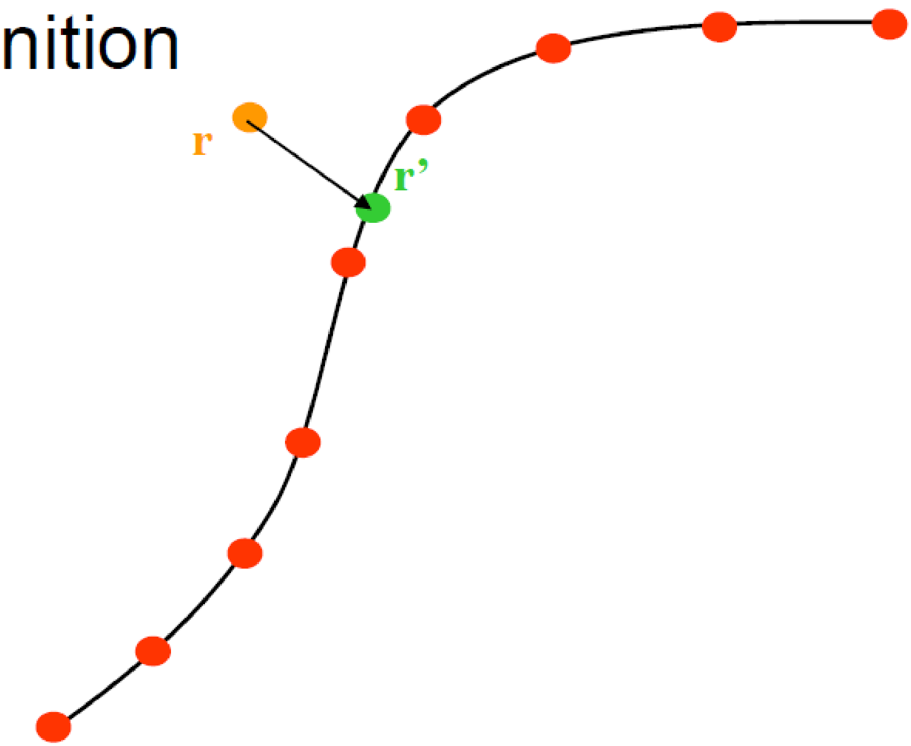
Projection

- Idea: Map space to surface
- Surface is defined as fixpoints of mapping



Surface definition

- Projection procedure (Levin)
  - Local polyonmial approximation
    - Inspired by differential geometry
  - “Implicit” surface definition
  - Infinitely smooth &
  - Manifold surface



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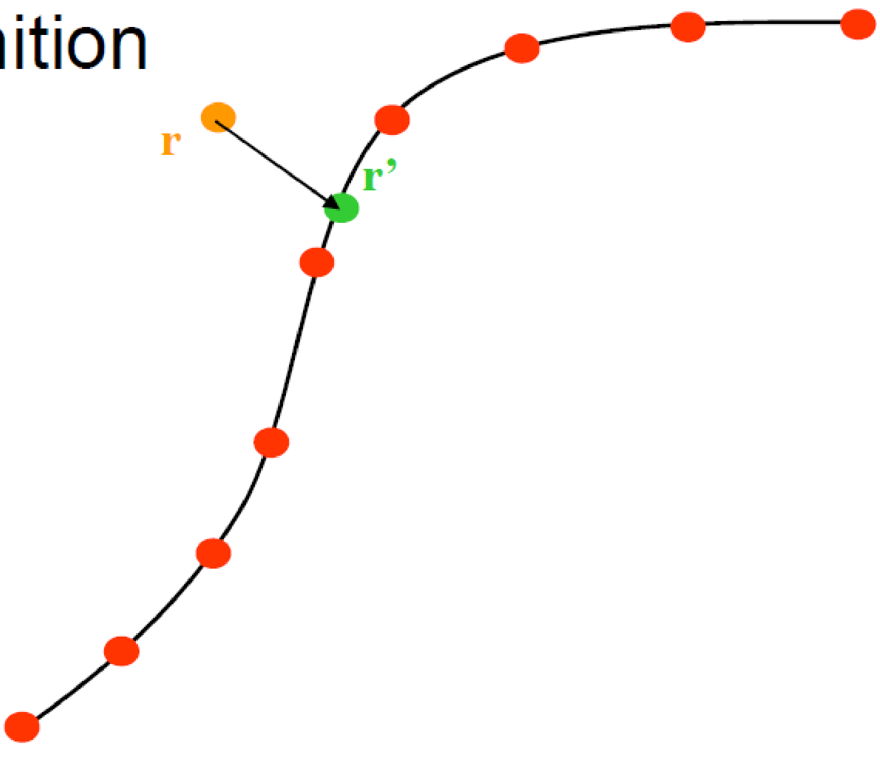
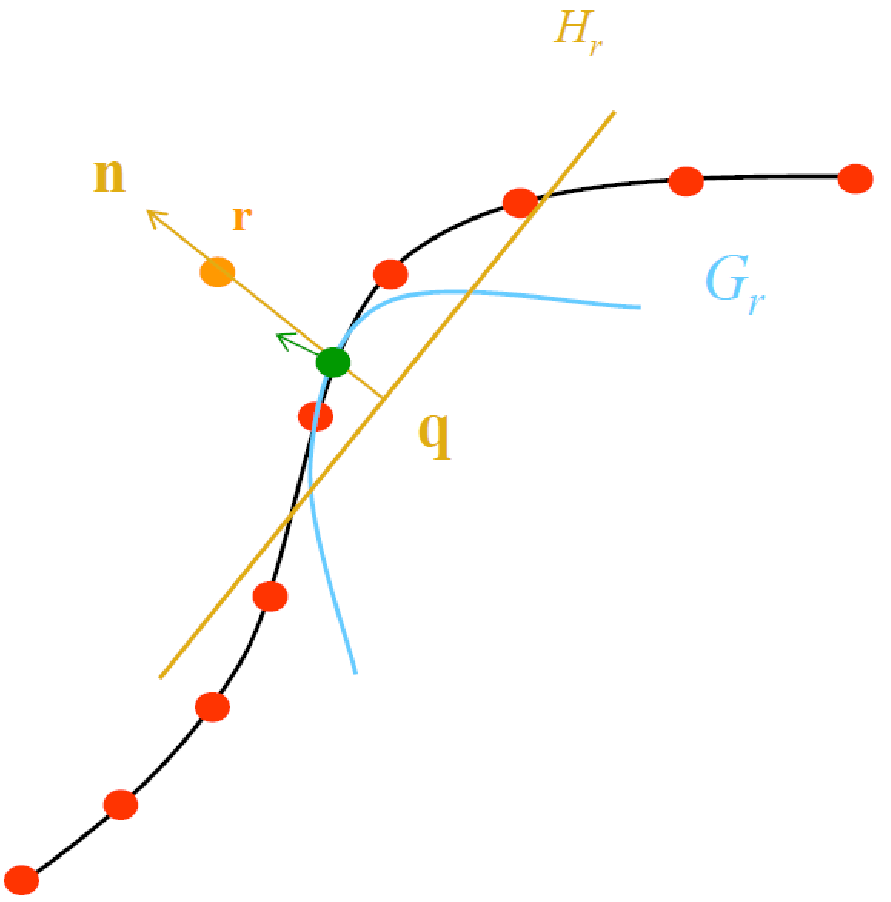


Surface definition

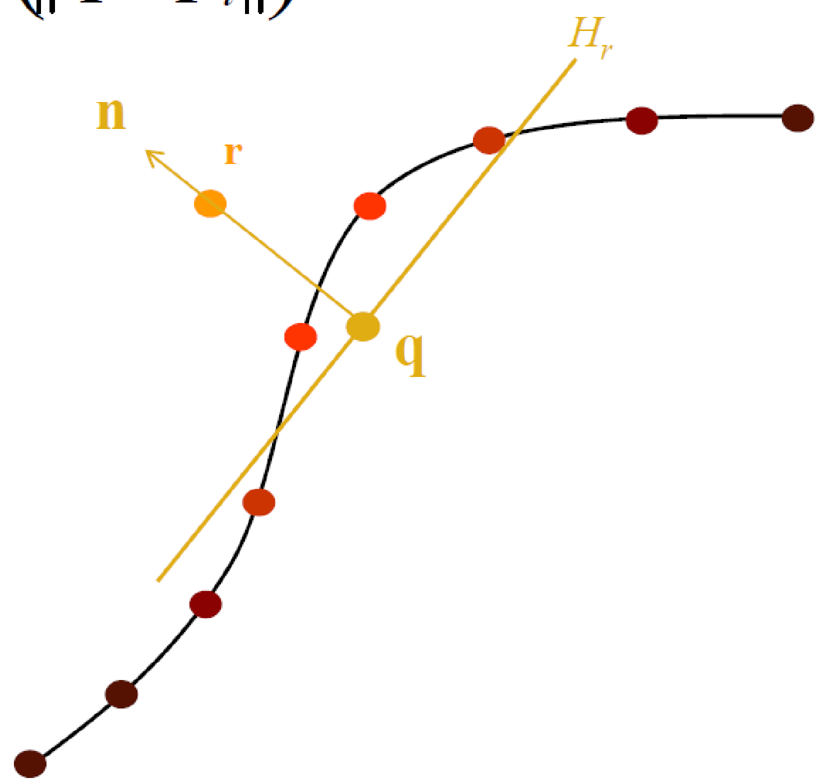
- Constructive definition
  - Input point $\mathbf{r}$
  - Compute a local reference plane $H_{\mathbf{r}}=\langle \mathbf{q},\mathbf{n}\rangle$
  - Compute a local polynomial over the plane $G_{\mathbf{r}}$
  - Project point $\mathbf{r}'=G_{\mathbf{r}}(0)$
  - Estimate normal



Local reference surface

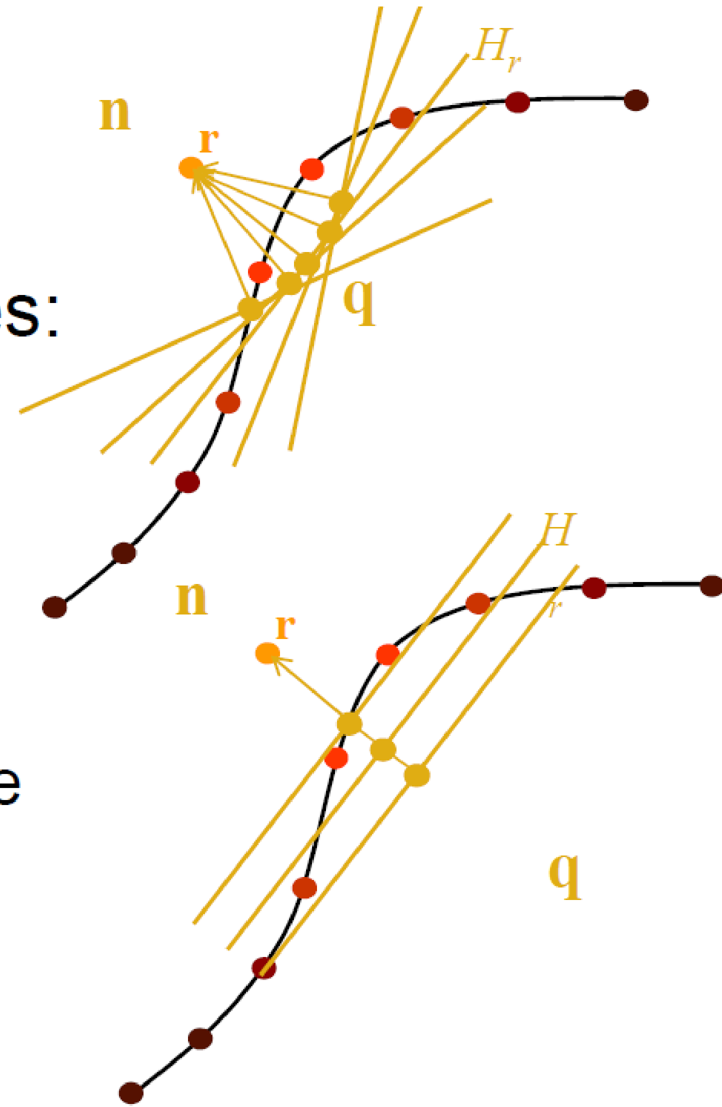
- Find plane $H_r = \langle \mathbf{q}, \mathbf{n} \rangle + D$
  - $\min_{\mathbf{q}, \|\mathbf{n}\|=1} \sum_i \langle \mathbf{q} - \mathbf{p}_i, \mathbf{n} \rangle^2 \theta(\|\mathbf{q} - \mathbf{p}_i\|)$
  - $\theta(\mathrm{d}) = e^{d^2/h^2}$
    - h is feature size/ point spacing
  - H_r is independent of r's distance
  - Manifold property

Weight function based on distance to q, not r



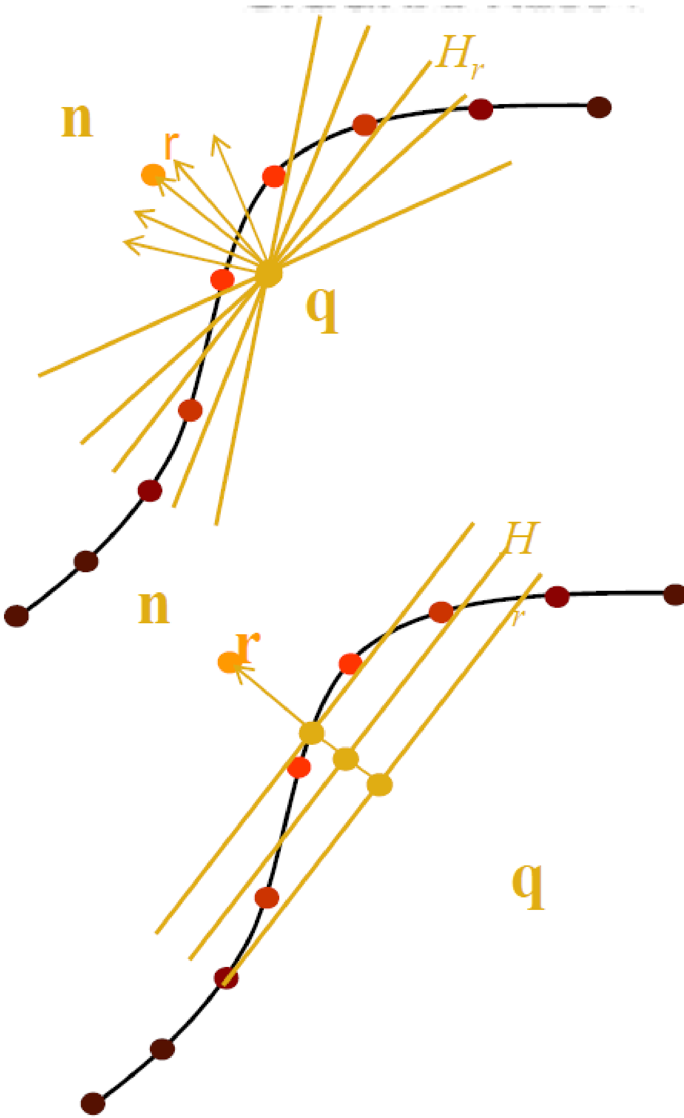
Local Reference Plane

- Computing reference plane
  - Non-linear optimization problem
- Minimize independent variables:
  - Over $\mathbf{n}$ for fixed distance $\|\mathbf{r}-\mathbf{q}\|$
  - Along n for fixed direction $\mathbf{n}$
  - $\mathbf{q}$ changes → the weights change
  - Only iterative solutions possible



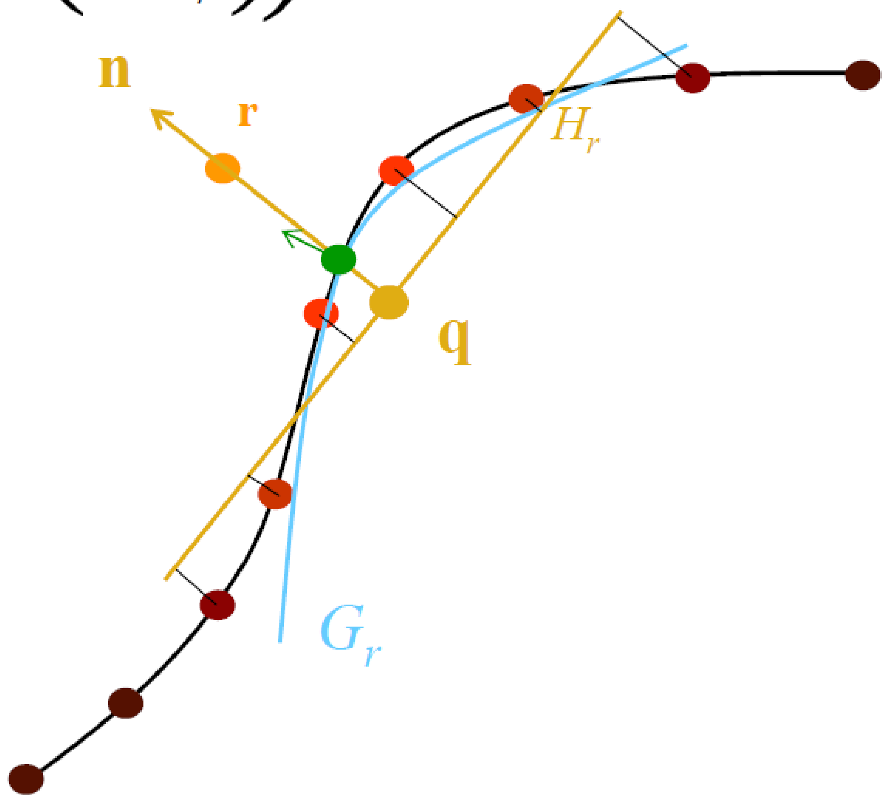
Local reference plane

- Practical computation
  - Minimize over $\mathbf{n}$ for fixed $\mathbf{q}$
    - Eigenvalue problem
  - Translate $\mathbf{q}$ so that
    $$\mathbf{r} = \mathbf{q} + \|\mathbf{r} - \mathbf{q}\|\mathbf{n}$$
    - Effectively changes $\|\mathbf{r} - \mathbf{q}\|$
  - Minimize along $\mathbf{n}$ for fixed direction $\mathbf{n}$
    - Exploit partial derivative



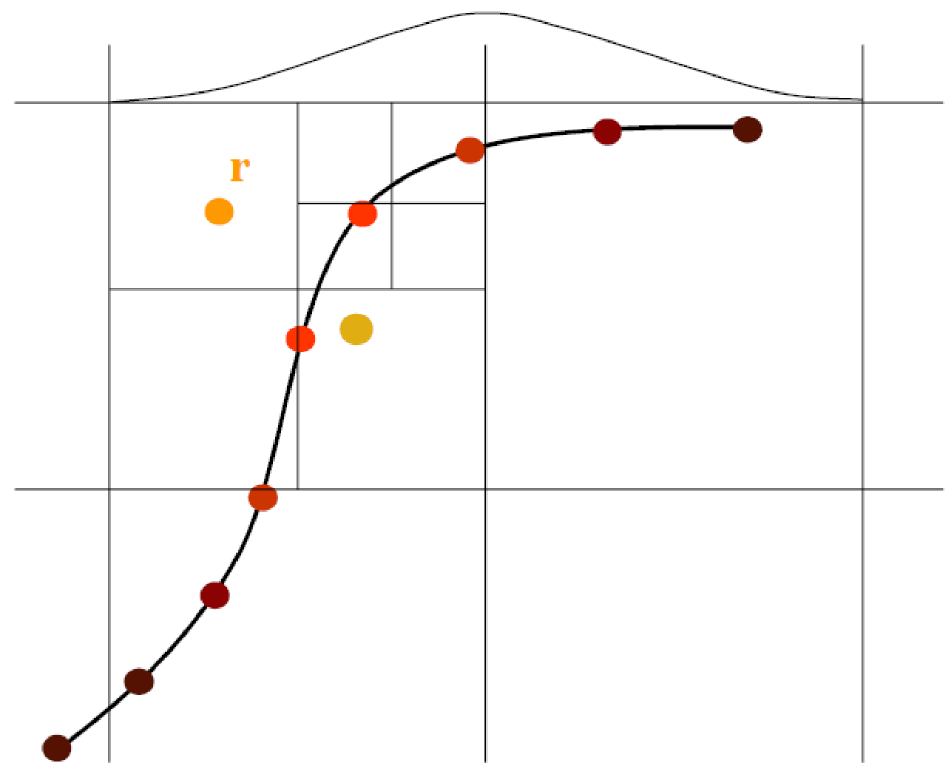
Rejecting the point

- MLS polyonomial over H_r
  - $\min\limits_{G \in \Pi_d} \sum\limits_i \left(\langle \mathbf{q} - \mathbf{p}_i, \mathbf{n} \rangle - G\left(\mathbf{p}_i \big|_{H_r} \right) \right)^2 \theta\left(\| \mathbf{q} - \mathbf{p}_i \| \right)$
  - LS problem
  - $\mathbf{r}' = G_r(0)$
  - Estimate normal



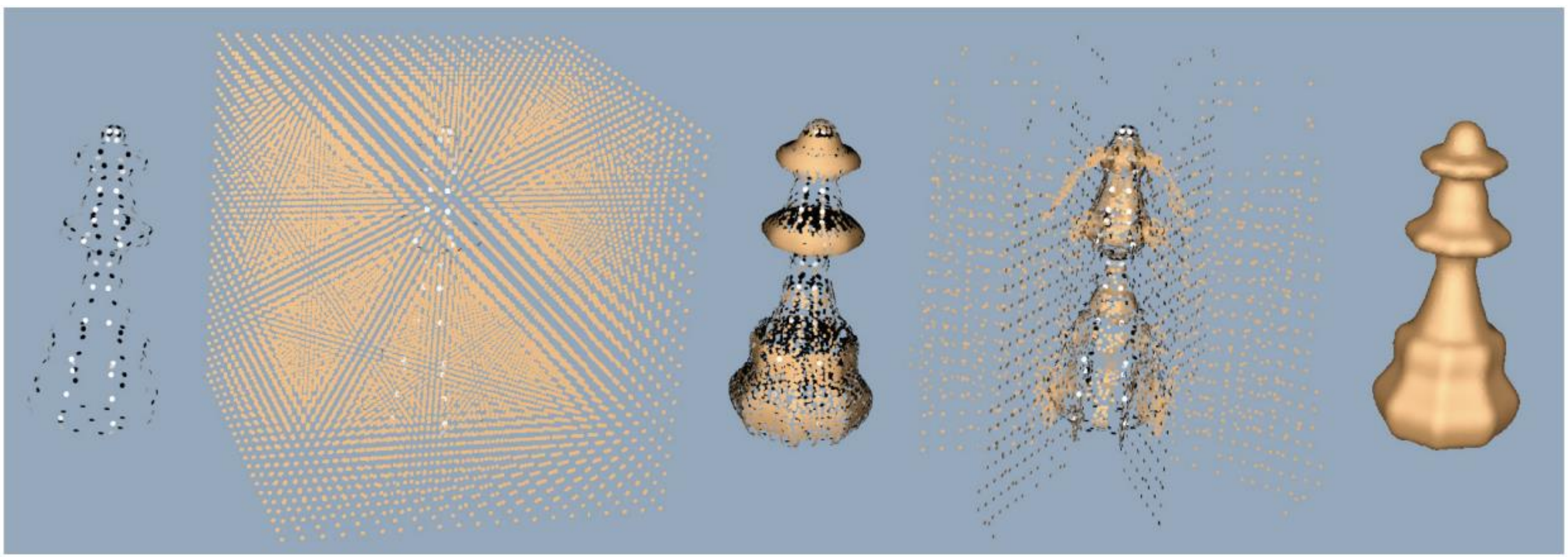
Spatial data structure

- Regular grid based on support of θ
  - Each point influences only 8 cells
- Each cell is an octree
  - Distant octree cells are approximated by one point in center of mass



Defining point-set surfaces

[Amenta and Kil 05]



Poisson Surface Reconstruction

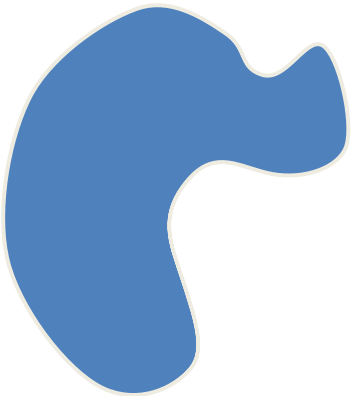
Poisson surface reconstruction

- Michael Kazhdan, M. Bolitho, and H. Hoppe, SGP 2006
- Source Code available at: http://www.cs.jhu.edu/~misha/
- Implementation included in Meshlab
- Relevant works
  - Poisson mesh editing [SIGGRAPH 2004, SGP 2004]
  - Poisson image editing [SIGGRAPH 2003]

Poisson surface reconstruction

- Indicator Function
  - reconstruct the surface by solving for the indicator function of the shape
  - Assume normal as inputs

$$\chi_M(p) = \begin{cases} 1 & \text{if } p \in M \\ 0 & \text{if } p \notin M \end{cases}$$

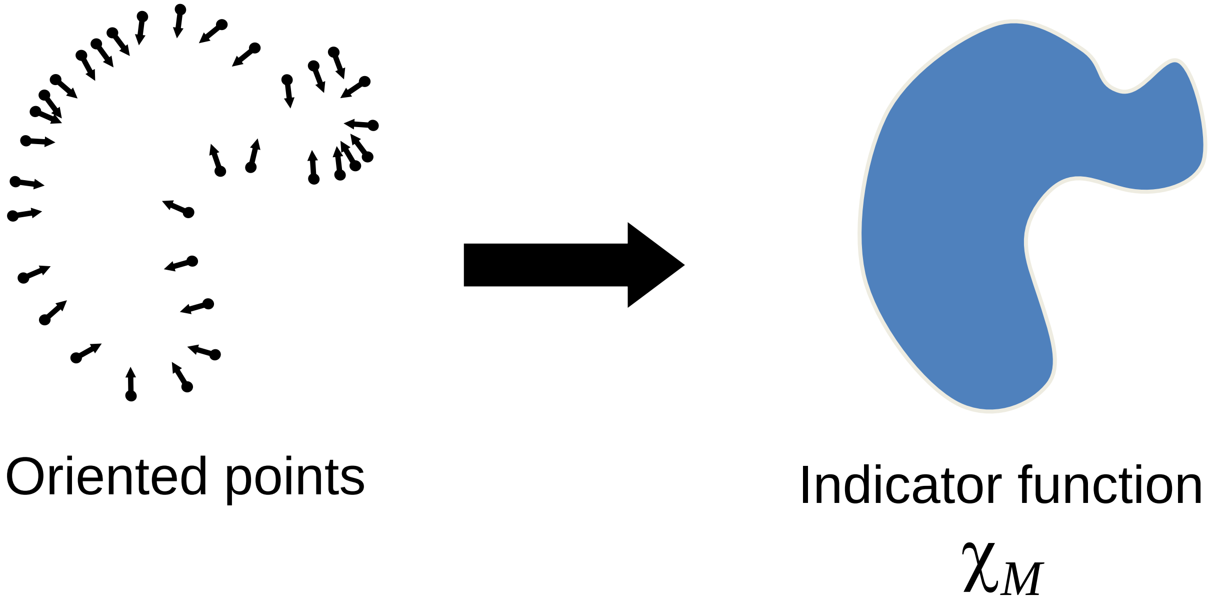


Indicator function

χ_M

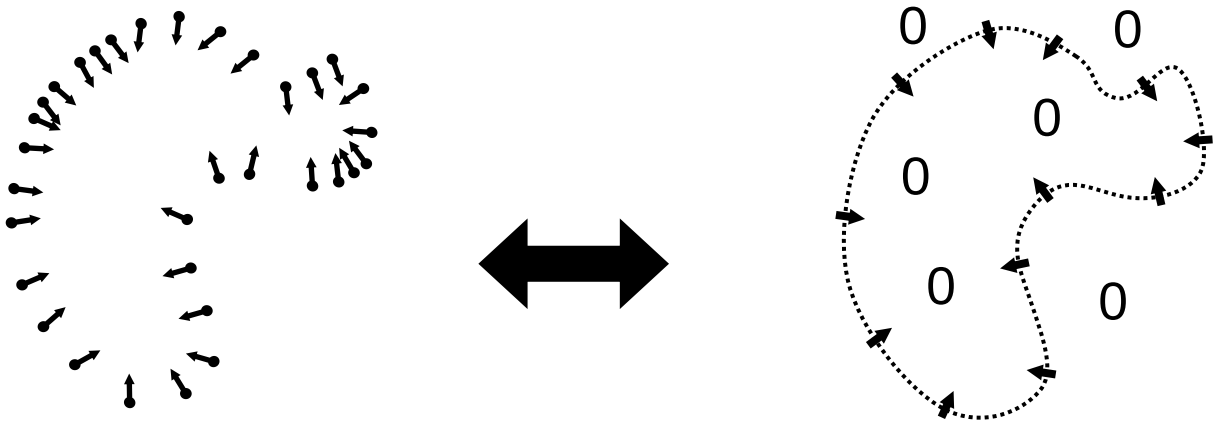
Problem

- Fit the indicator function to a set of oriented normal
  - Fitting should be robust



Gradient Relationship

- There is a relationship between the normal field and gradient of indicator function



Oriented points

Indicator gradient

$\nabla \chi_M$

Integration

- Represent the points by a vector field
- Find the function χ whose gradient best approximates $\vec{V}$:

$$\min_{\chi} \left\| \nabla \chi - \vec{V} \right\|$$

Integration

- Represent the points by a vector field
- Find the function χ whose gradient best approximates $\vec{V}$:

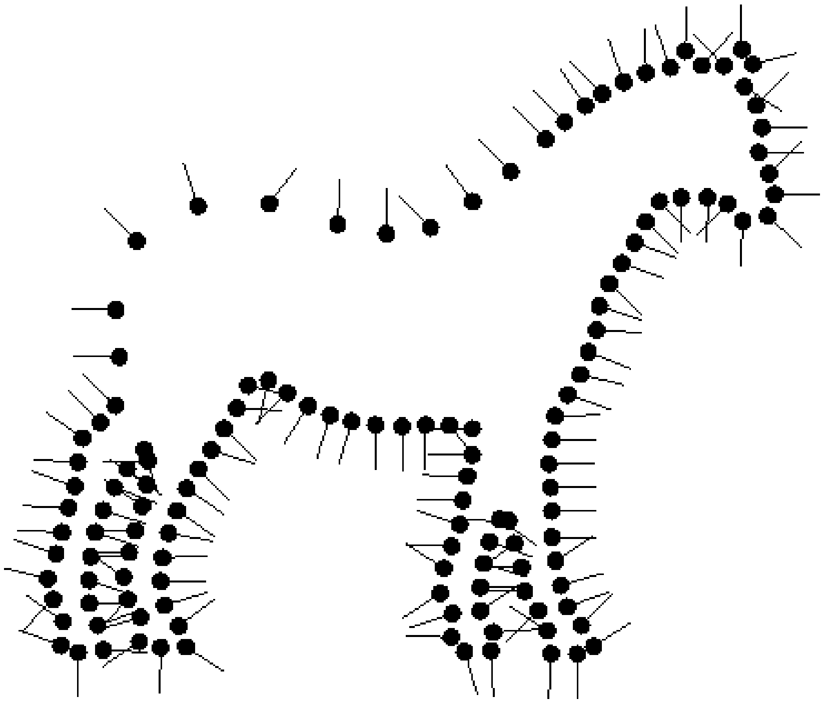
$$\min_{\chi} \left\| \nabla \chi - \vec{V} \right\|$$

- Applying the divergence operator, we can transform this into a Poisson problem:

$$\nabla \cdot (\nabla \chi) = \nabla \cdot \vec{V} \quad \Leftrightarrow \quad \Delta \chi = \nabla \cdot \vec{V}$$

Implementation

- Given the Points:
  - Set octree
  - Compute vector field
  - Compute indicator function
  - Extract iso-surface



Implementation: Adapted Octree

- Given the Points:
  - Set octree
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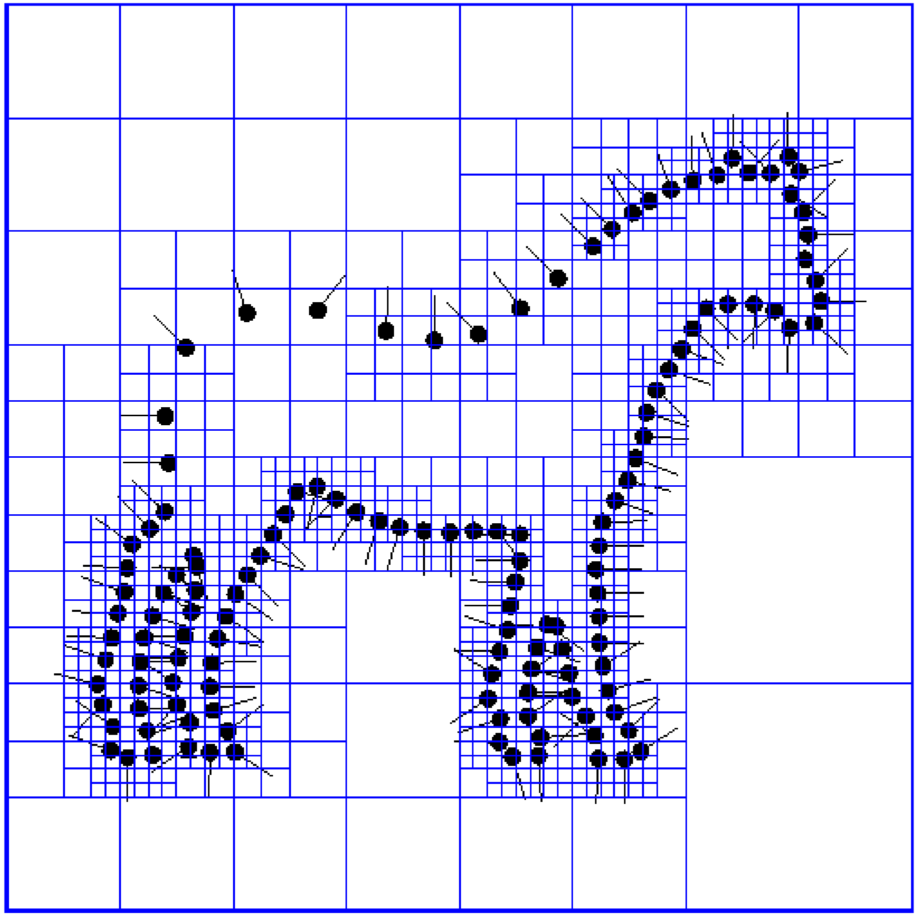
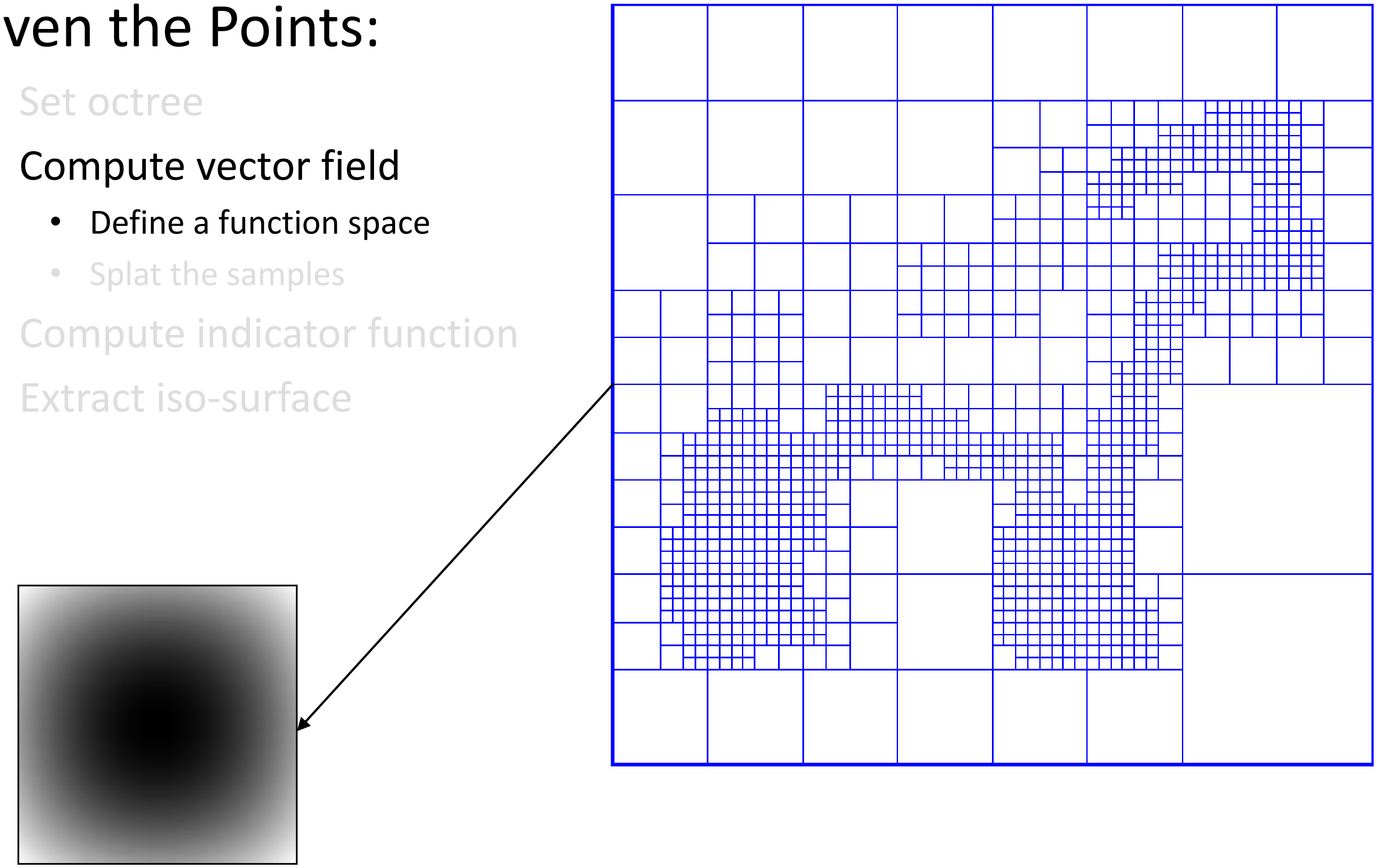


Implementation: Vector Field

- Given the Points:
  - Set octree
  - Compute vector field
    - Define a function space
    - Splat the samples
  - Compute indicator function
  - Extract iso-surface



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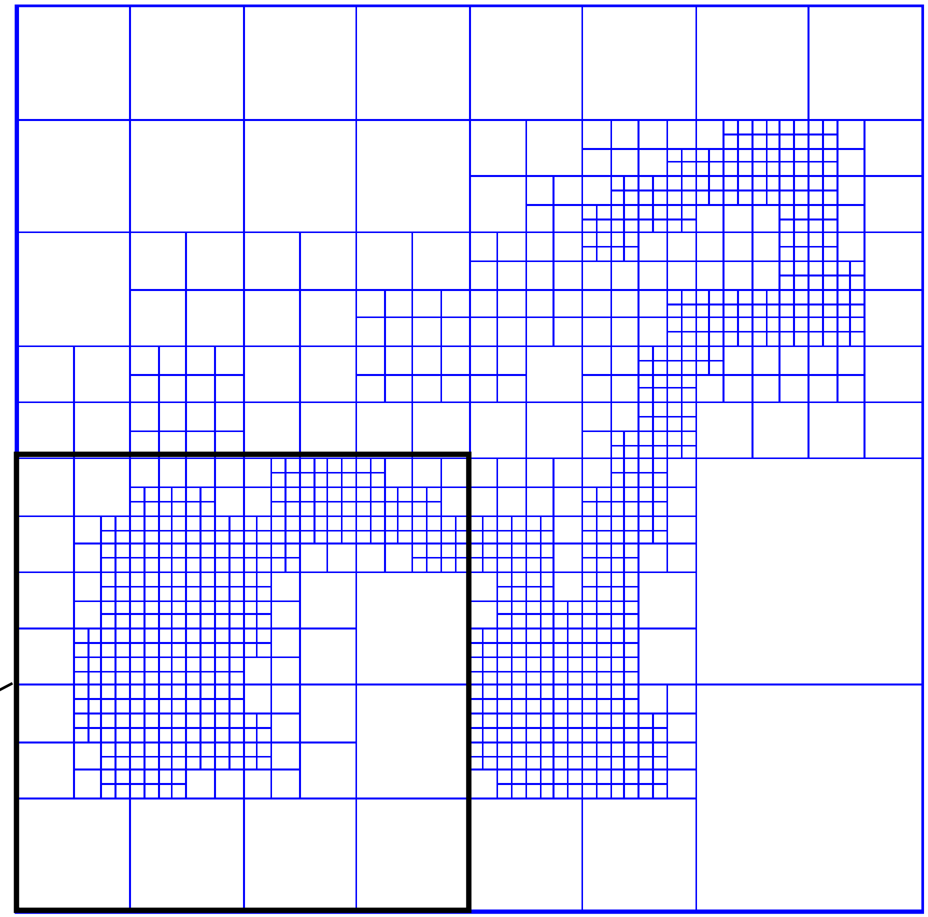

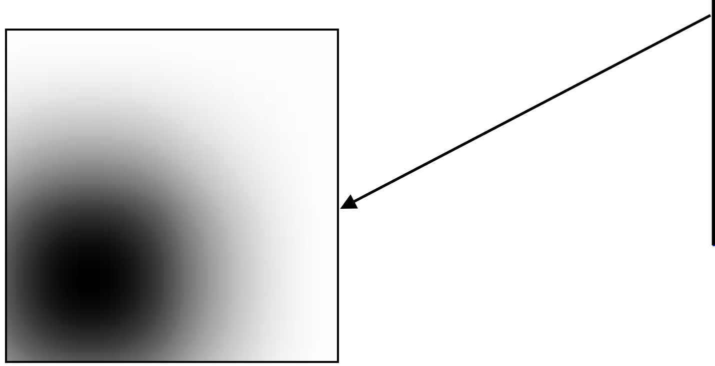

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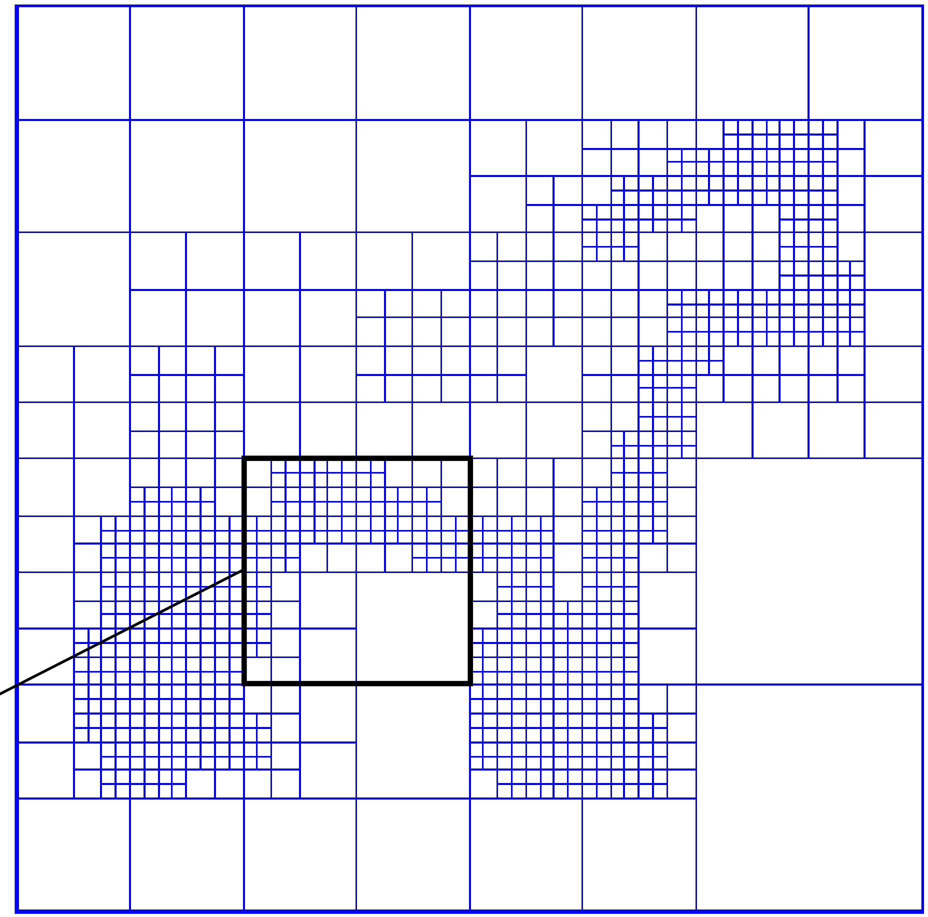

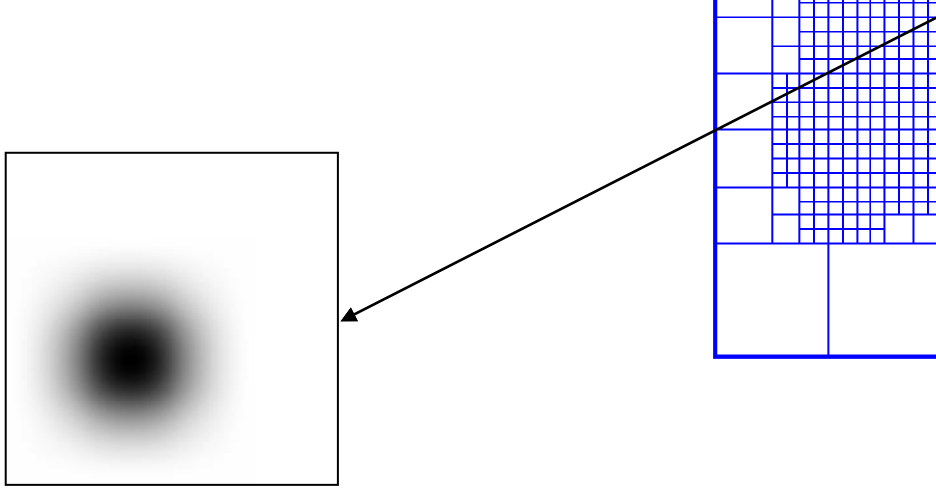

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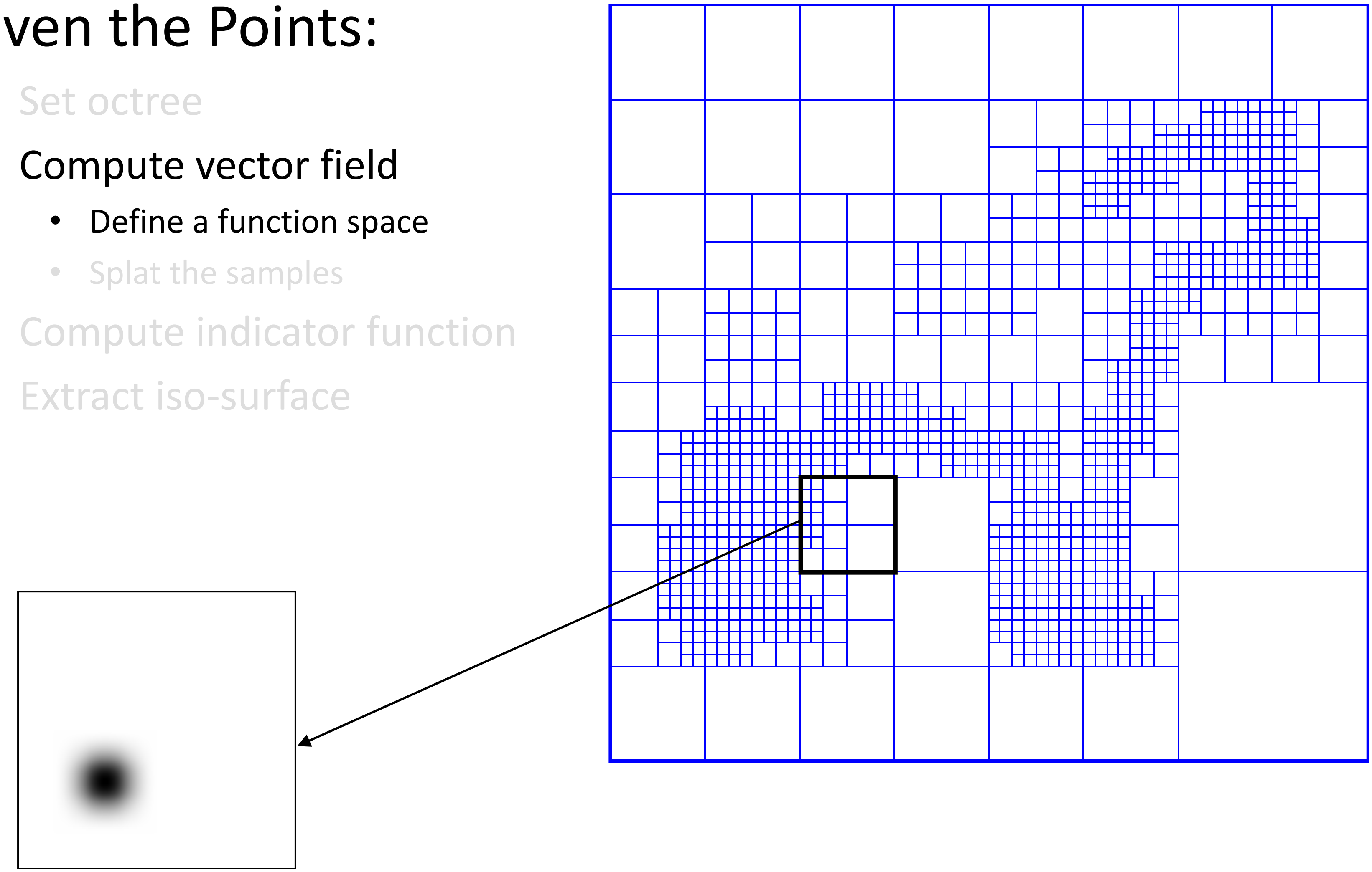

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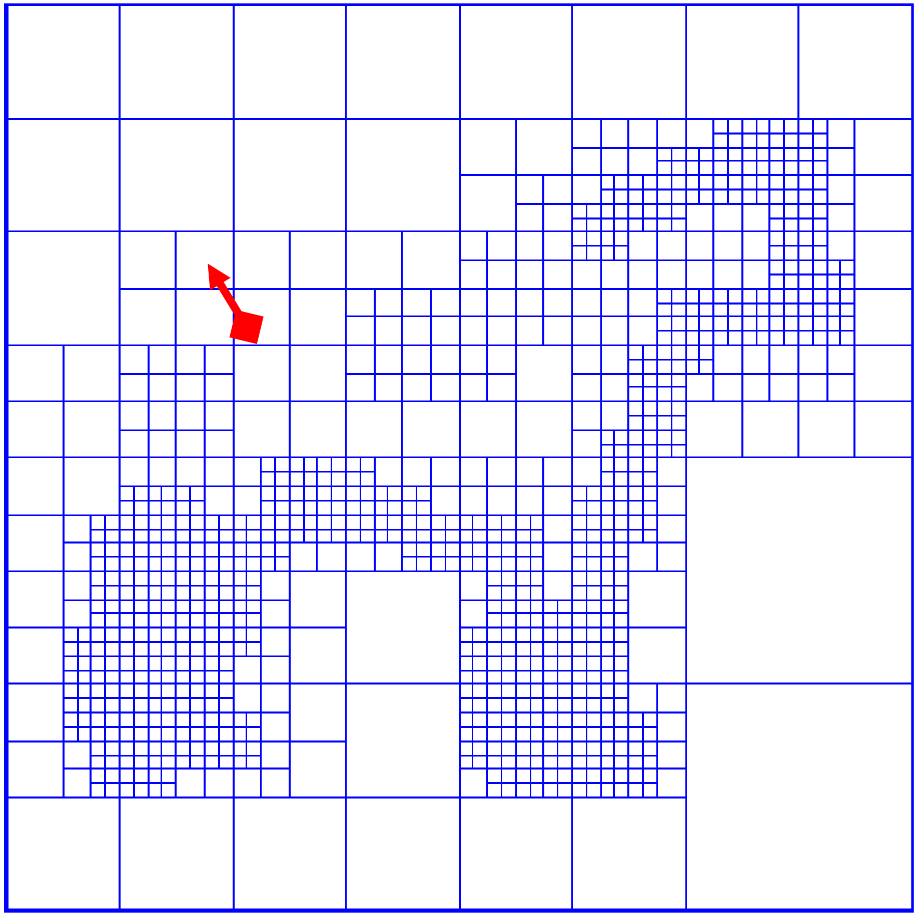

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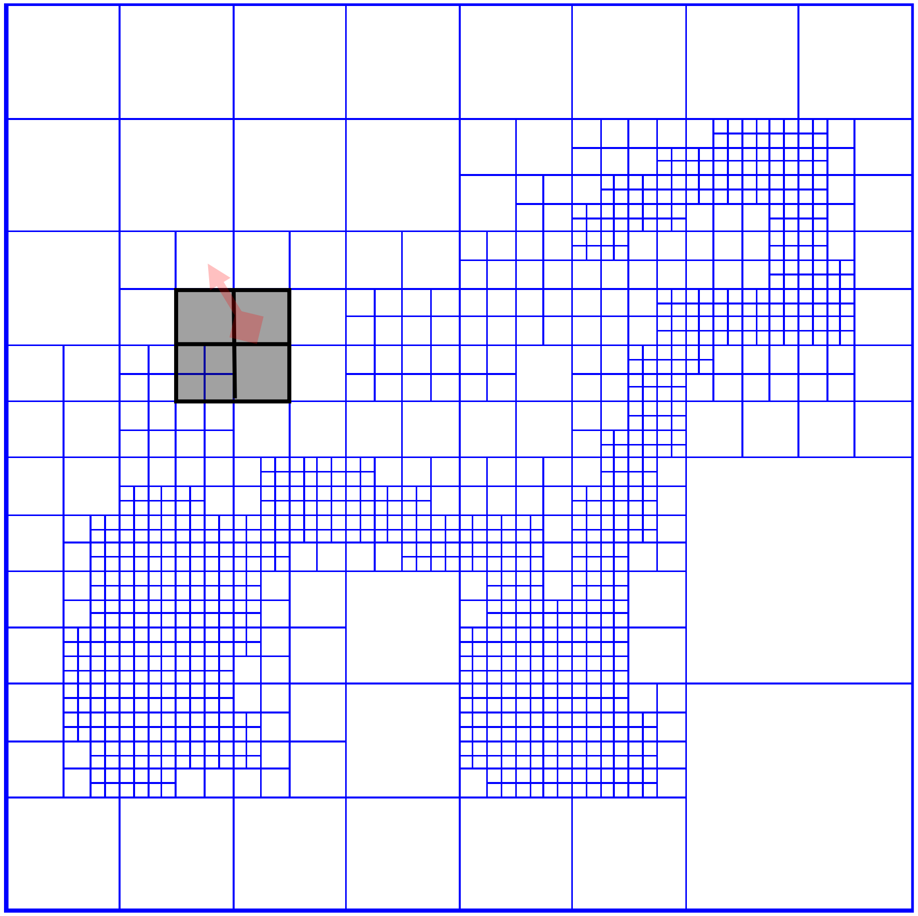

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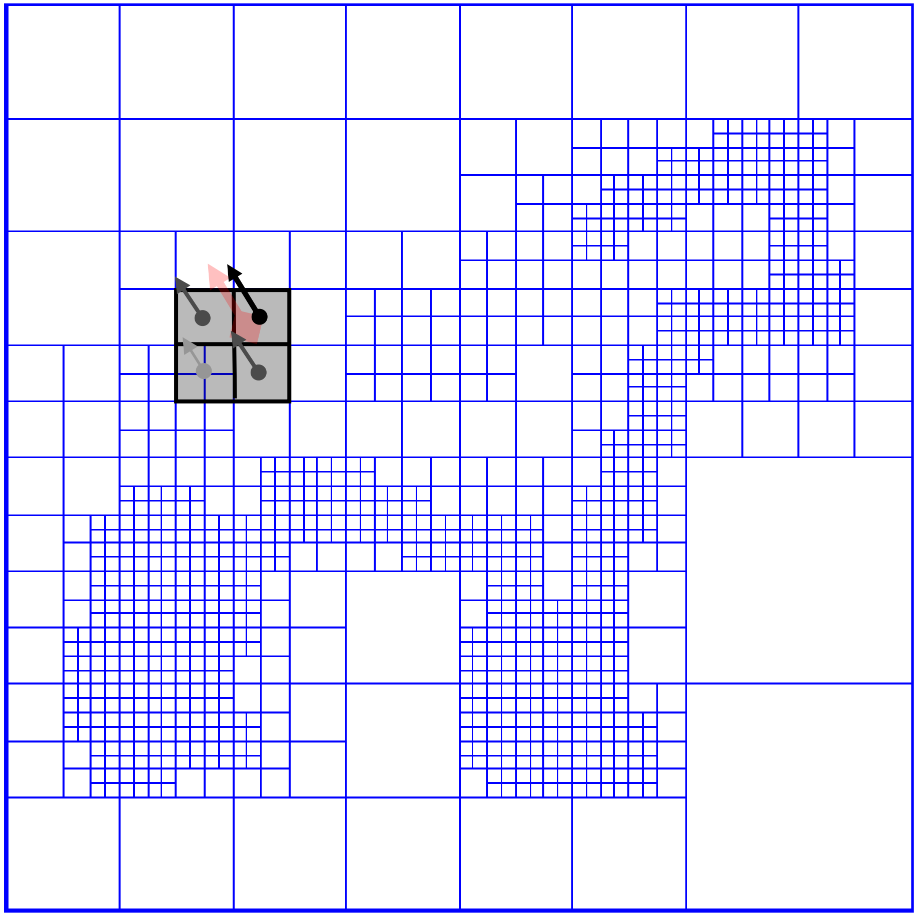
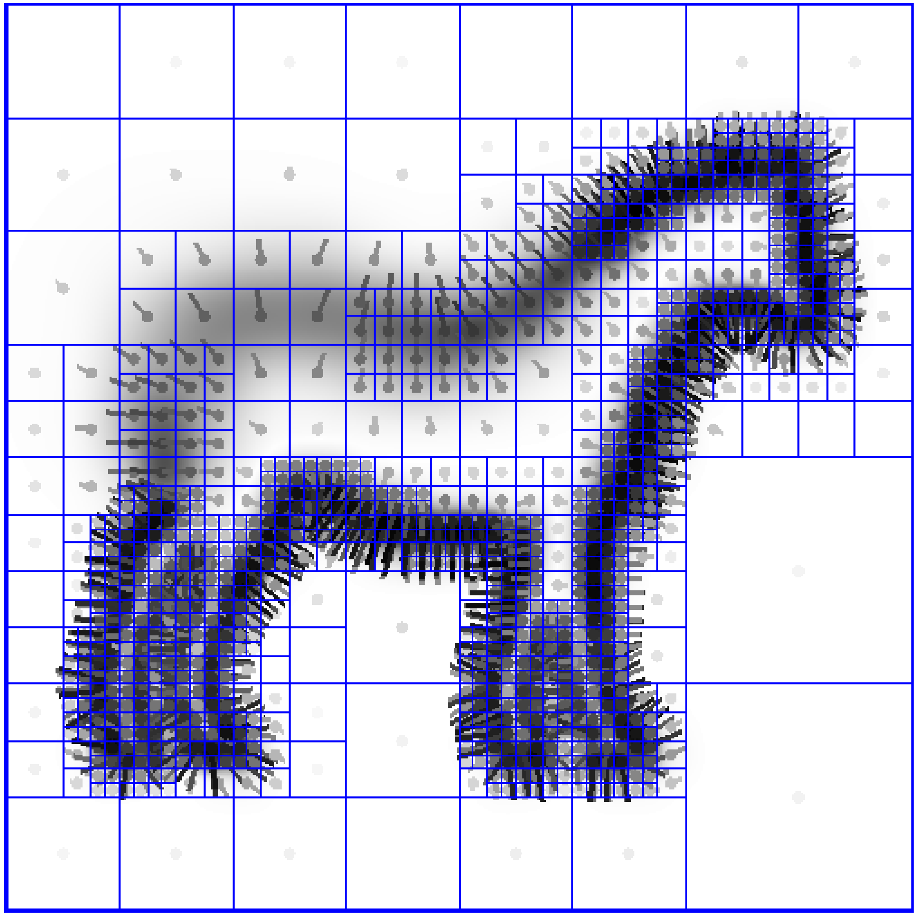
  - Extract iso-surface



Implementation: Vector Field

- Given the Points:
  - Set octree
  - Compute vector field
    - Define a function space
    - Splat the samples
  - Compute indicator function
  - Extract iso-surface



Implementation: Vector Field

- Given the Points:
  - Set octree
  - Compute vector field
  - Compute indicator function
    - Compute divergence
    - Solve Poisson equation
  - Extract iso-surface

Implementation: Vector Field

- Given the Points:
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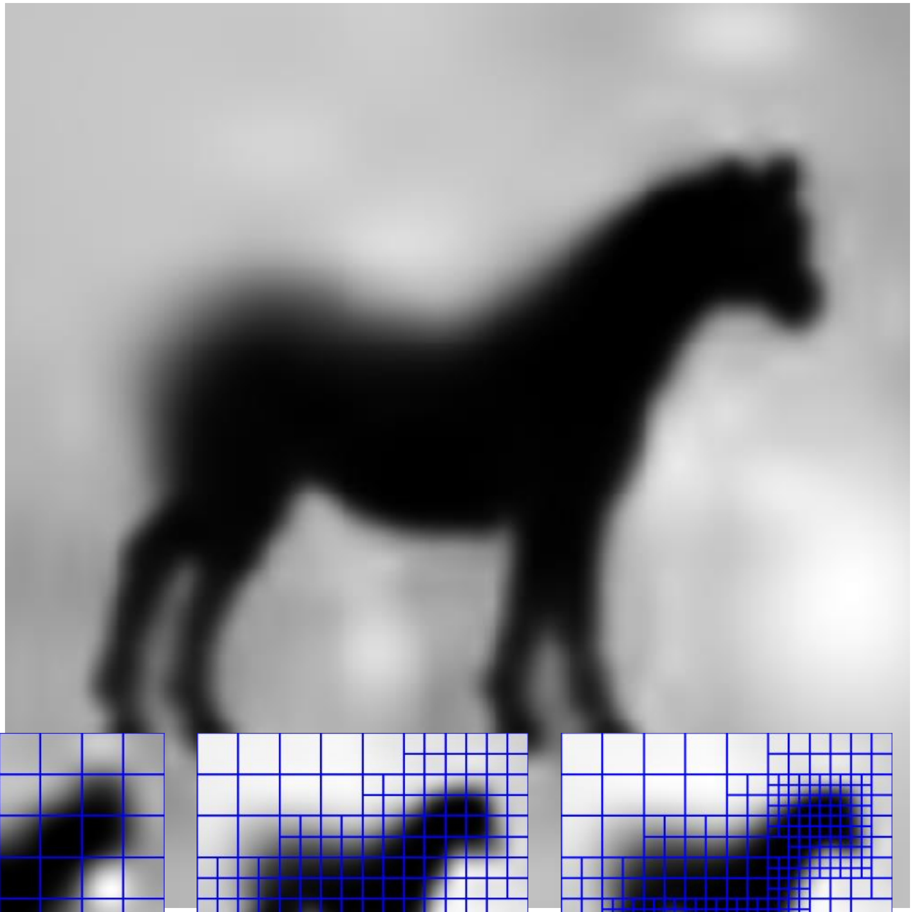

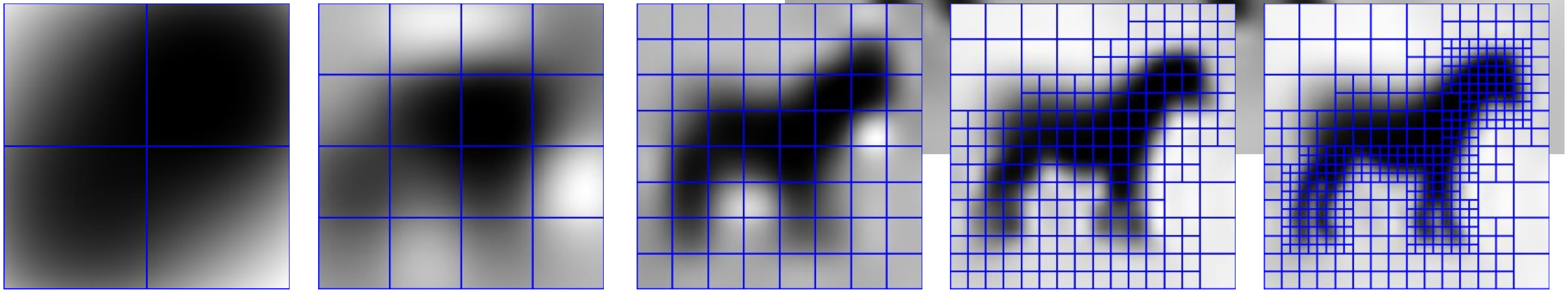

Implementation: Vector Field

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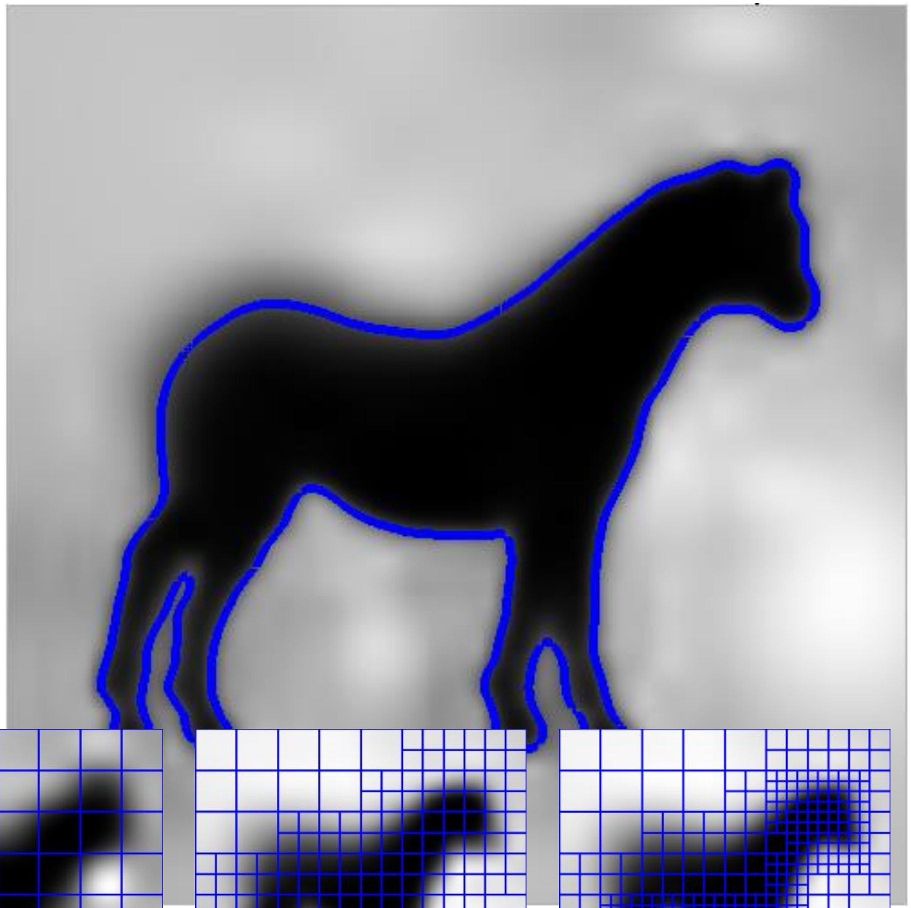
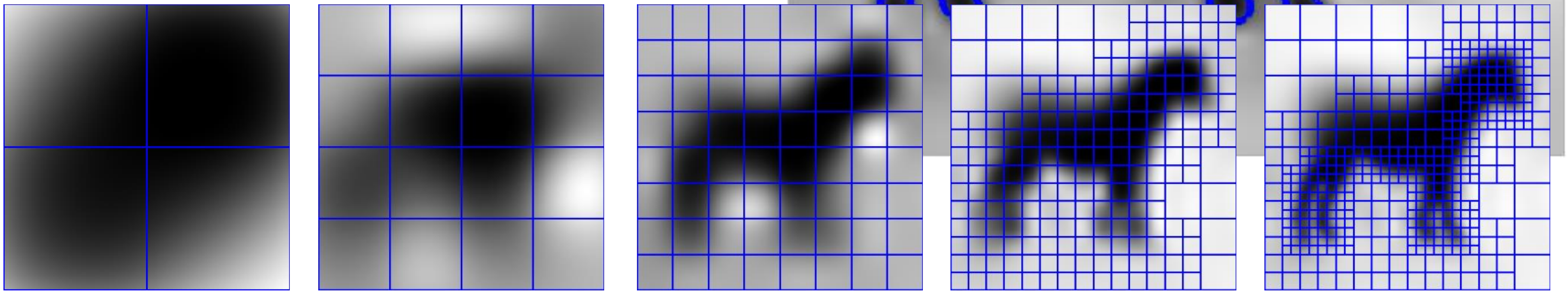
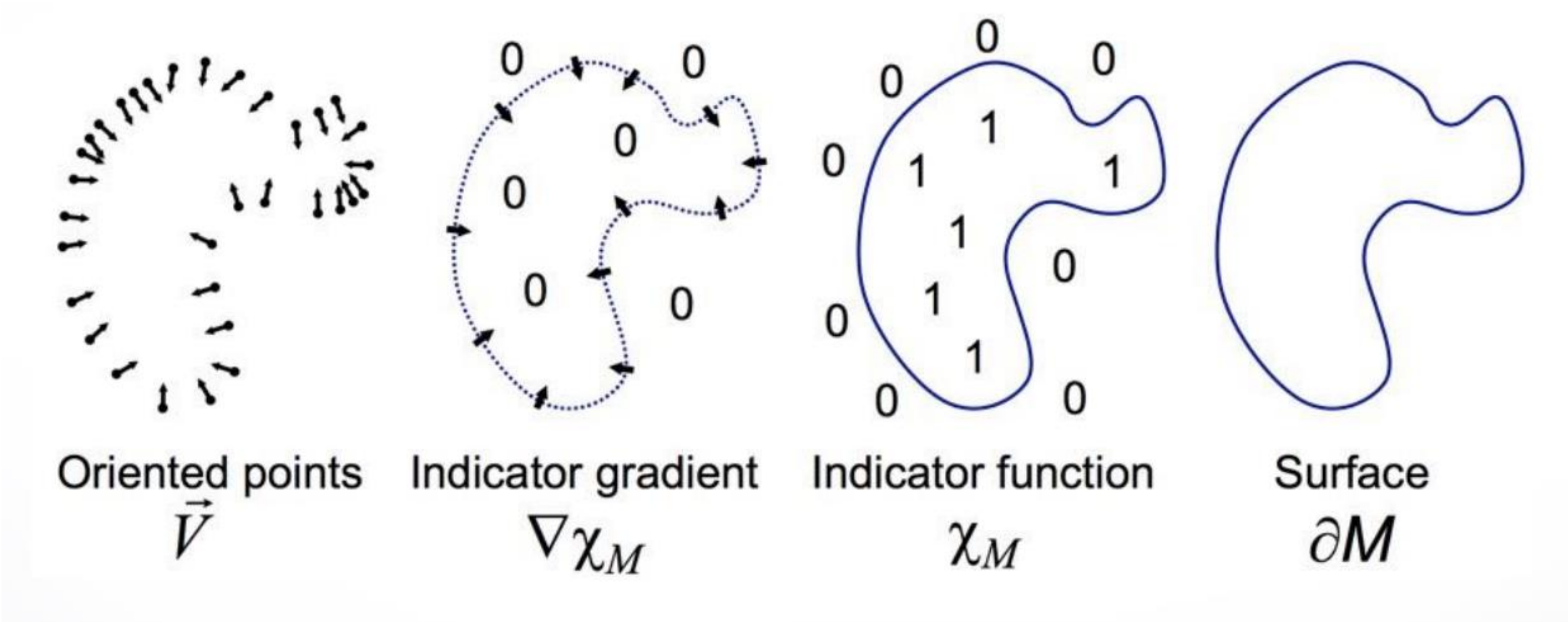




Summary


0
0
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0
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0
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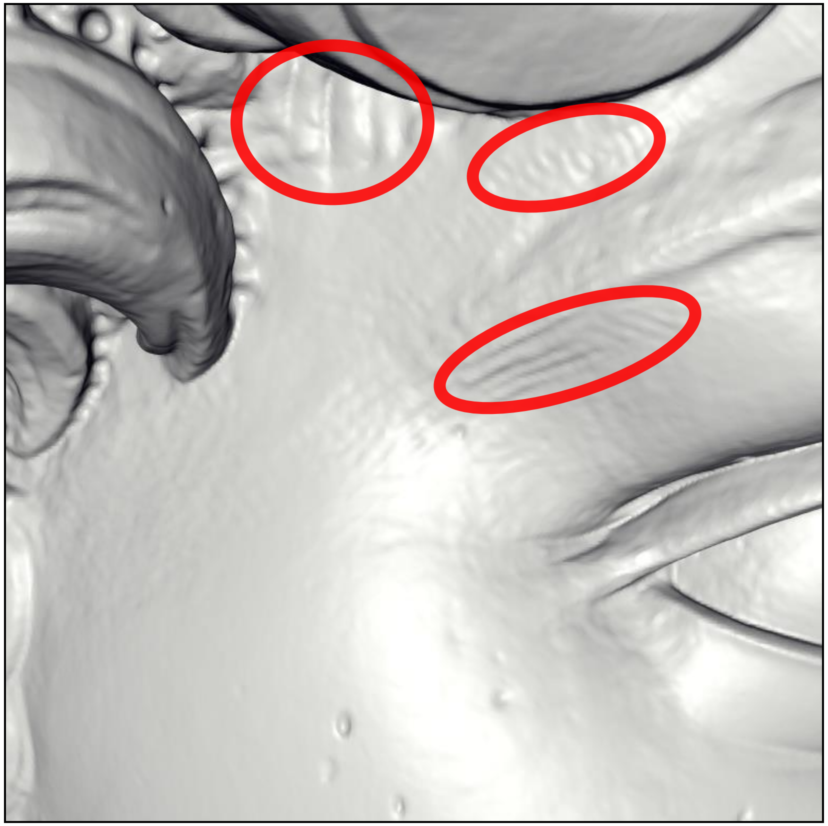
0
0
1
1
1
1
1
1
0
0
0
0
Oriented points
$\vec{V}$
Indicator gradient
$\nabla \chi_M$
Indicator function
χ_M
Surface
∂M

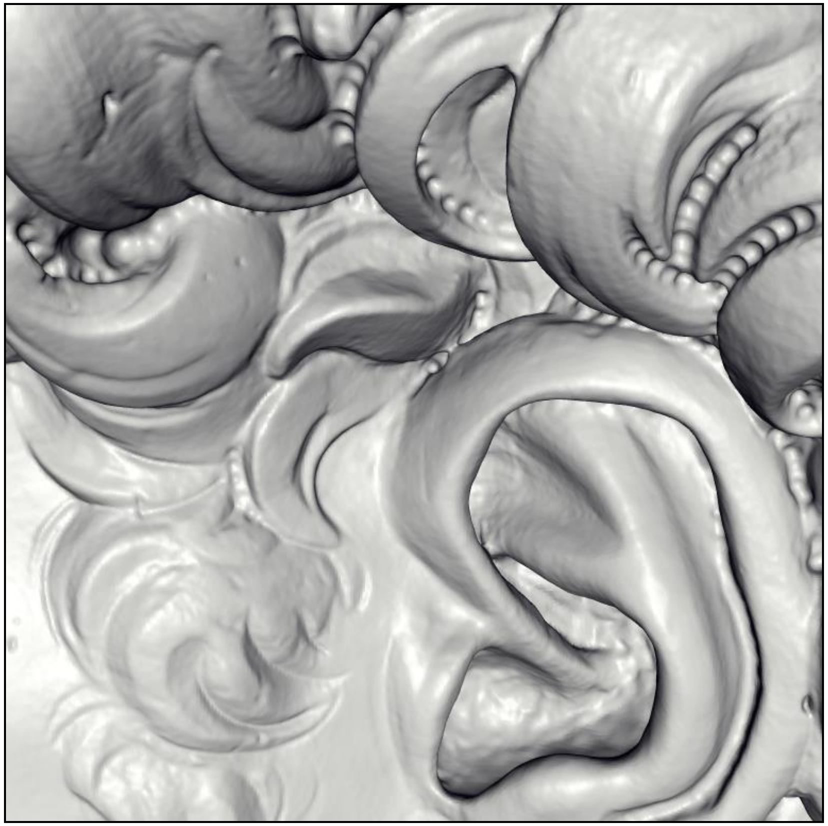
Michelangelo’s David

- 215 million data points from 1000 scans
- 22 million triangle reconstruction
- Compute Time: 2.1 hours
- Peak Memory: 6600MB

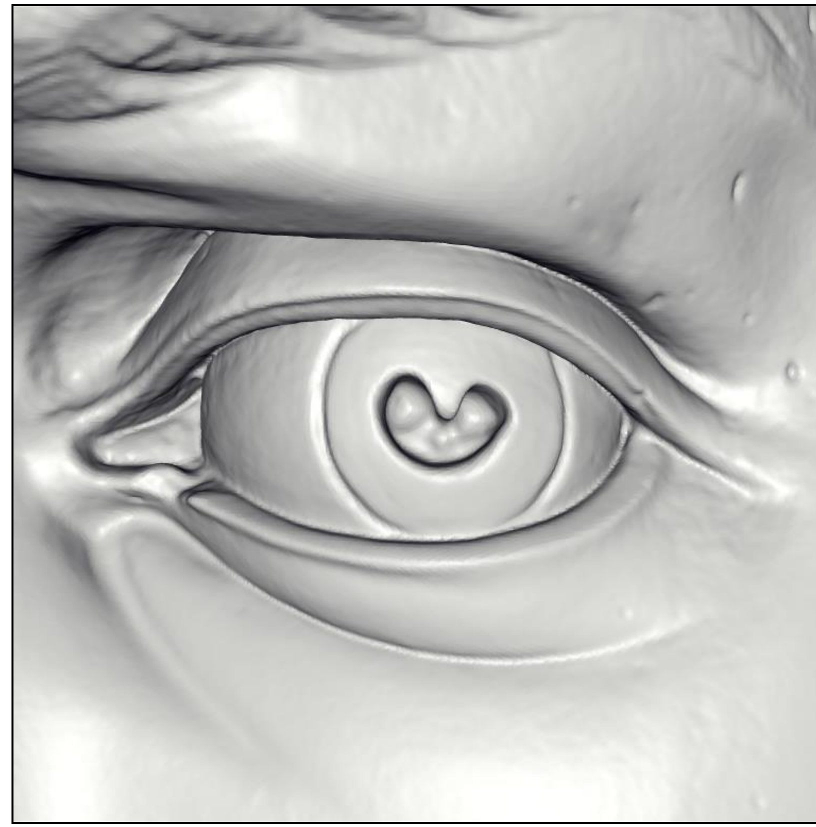
David – Chisel marks



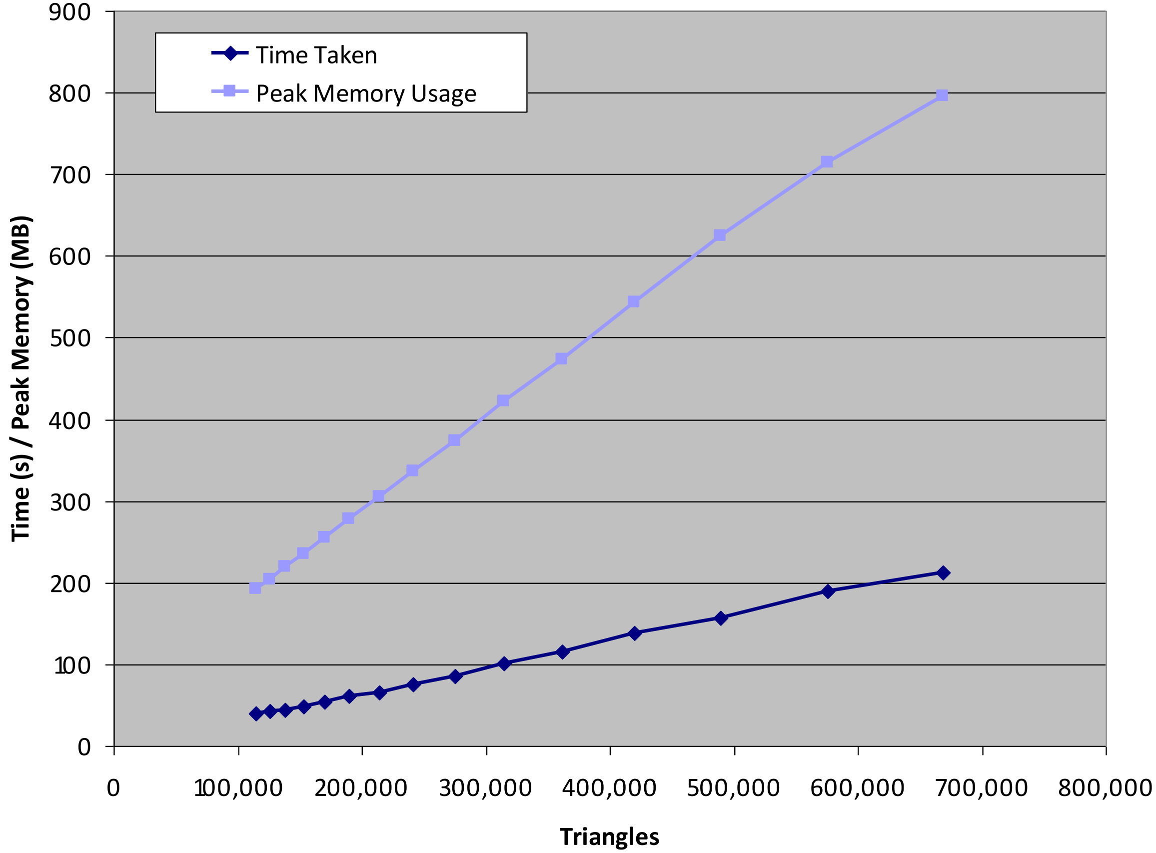
David – Drill Marks



David – Eye



Scalability – Buddha Model



Stanford Bunny

Power Crust

FastRBF

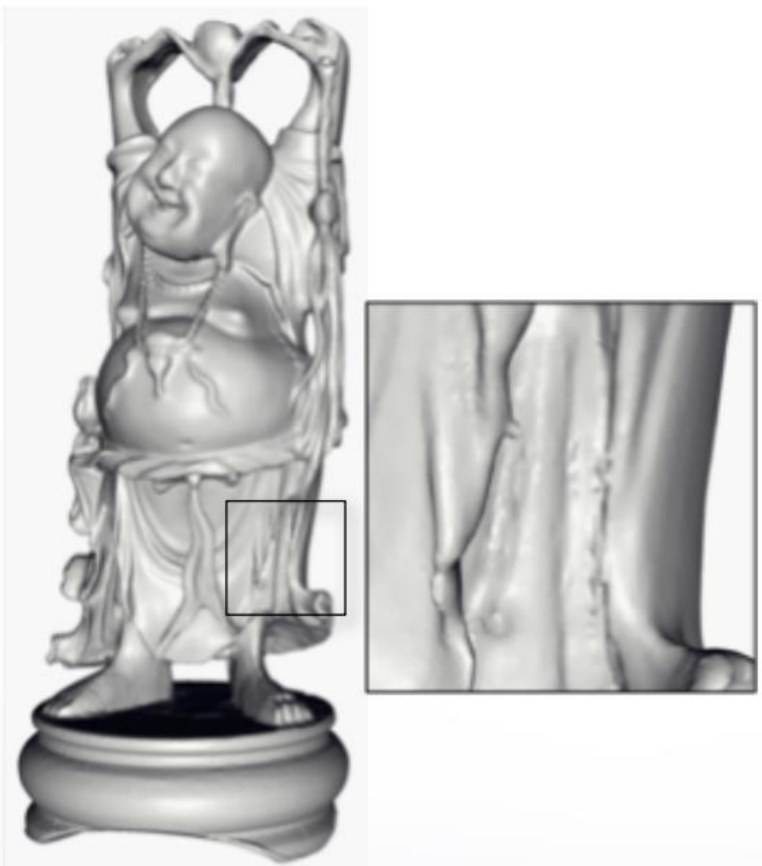
MPU

VRIP

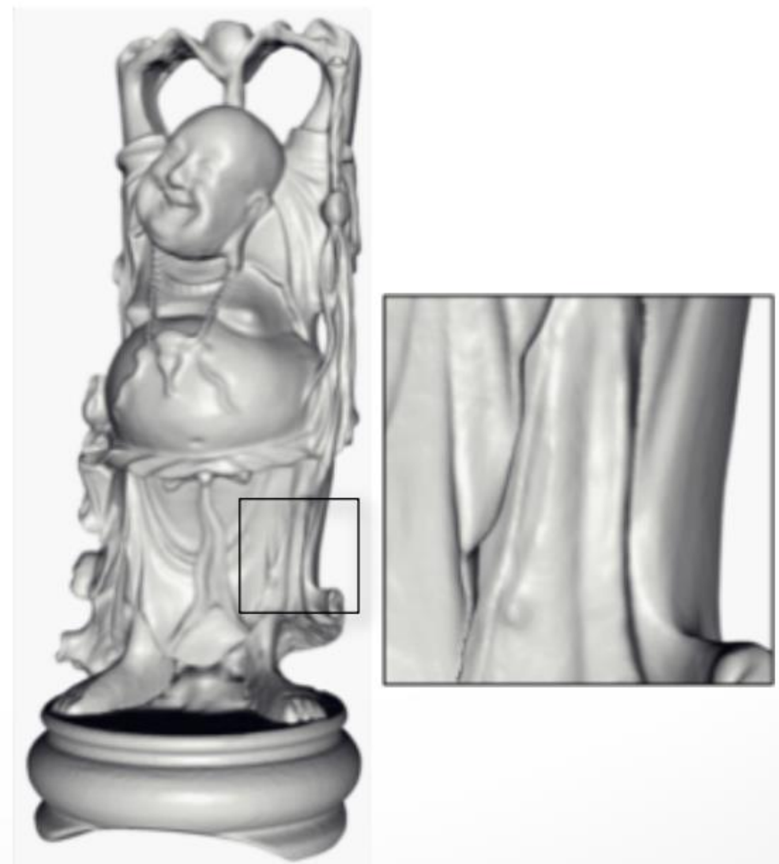
FFT Reconstruction

Poisson Reconstruction

VRIP Comparison



VRIP



Poisson Reconstruction

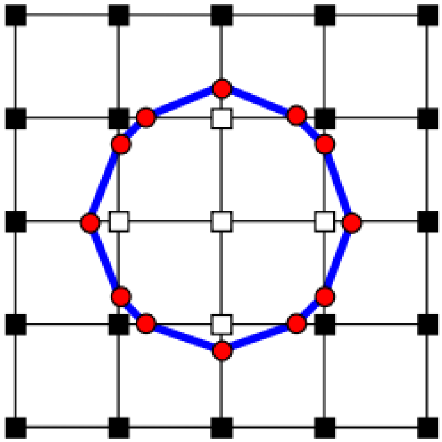
Neural Implicits

- We will cover this later

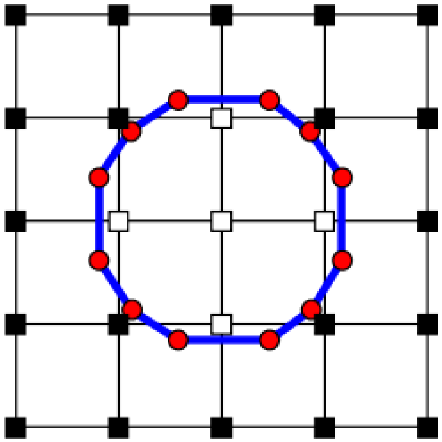
Meshing

Algorithms

- Primal methods
  - Marching Squares (2D), Marching Cubes (3D)
  - Placing vertices on grid edges



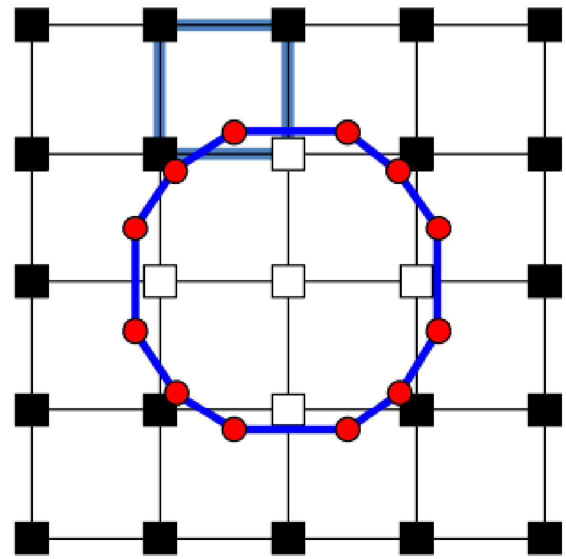
- Dual methods
  - Dual Contouring (2D,3D)
  - Placing vertices in grid cells



Slide Credit: Tao Ju

Dual Contouring (2D)

- For each grid cell with a sign change
  - Create one vertex
- For each grid edge with a sign change
  - Connect the two vertices in the adjacent cells with a line segment



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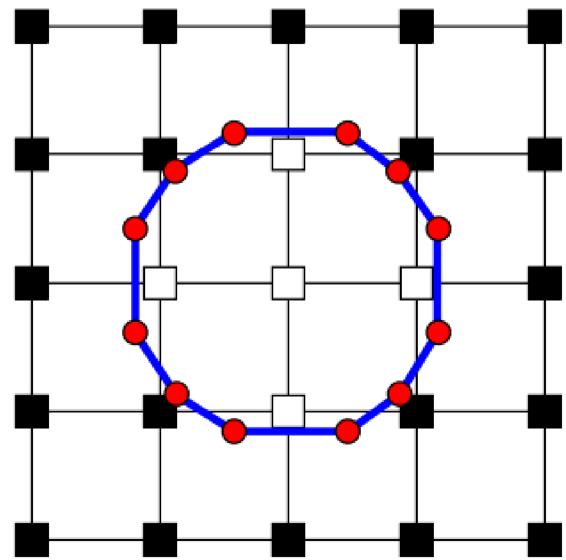
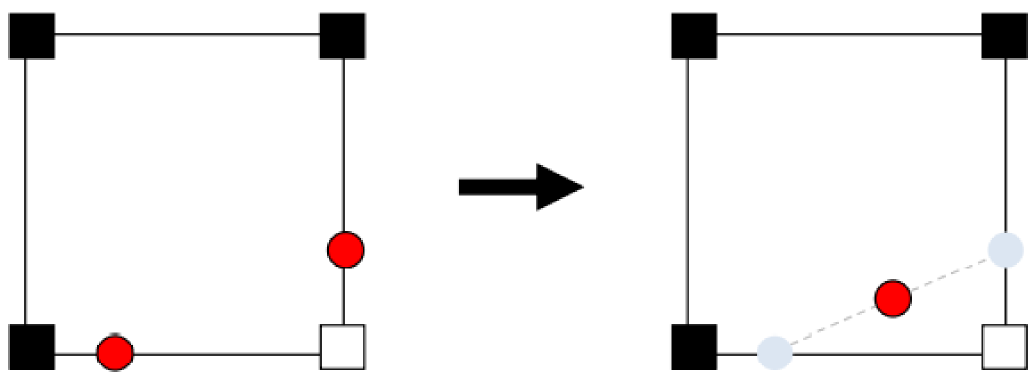


Slide Credit: Tao Ju

Dual Contouring (2D)

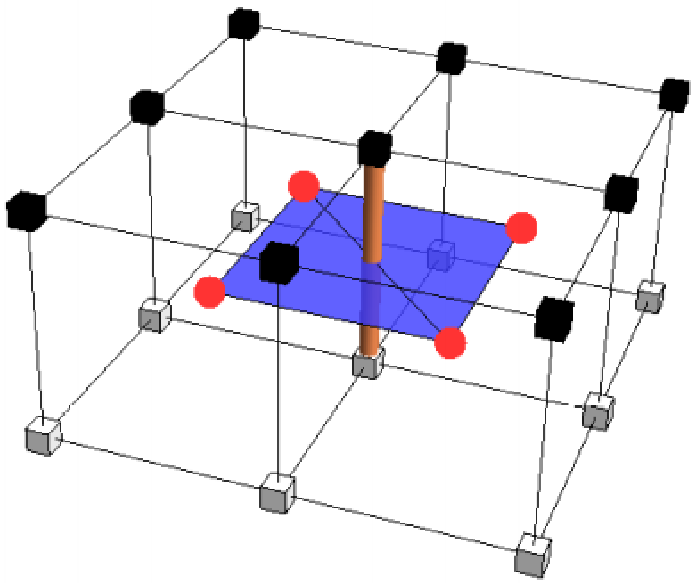
- Creating the vertex within a cell
  - Compute one point on each grid edge with a sign change (by linear interpolation)
    - There could be more than two sign-changing edges, so >2 points possible
  - Take the centroid of these points



Slide Credit: Tao Ju

Dual Contouring (3D)

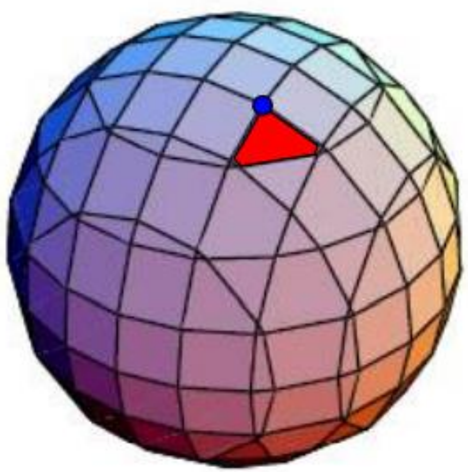
- For each grid cell with a sign change
  - Create one vertex (same way as 2D)
- For each grid edge with a sign change
  - Create a quad (or two triangles) connecting the four vertices in the adjacent grid cubes
  - No look-up table is needed!



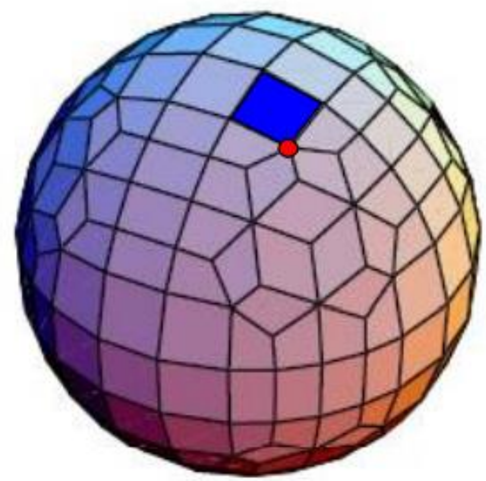
Slide Credit: Tao Ju

Duality

- The two outputs have a dual structure
  - Vertices and quads of Dual Contouring correspond (roughly) to un-triangulated polygons and vertices produced by Marching Cubes



Marching Cubes



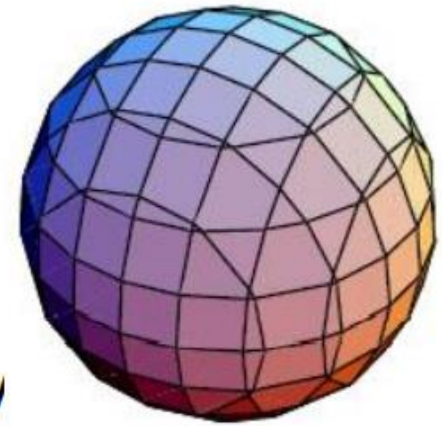
Dual Contouring

Slide Credit: Tao Ju

Slide Credit: Tao Ju

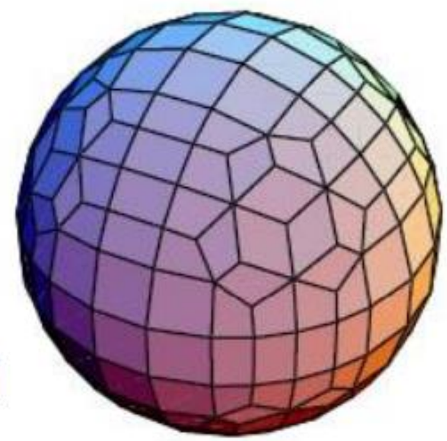
Primal vs. Dual

- Marching Cubes
  - ✓ Always manifold
  - ✗ Requires look-up table in 3D
  - ✗ Often generates thin and tiny poly



Marching Cubes

- Dual Contouring
  - ✗ Can be non-manifold
  - ✓ No look-up table needed
  - ✓ Generates better-shaped polygon

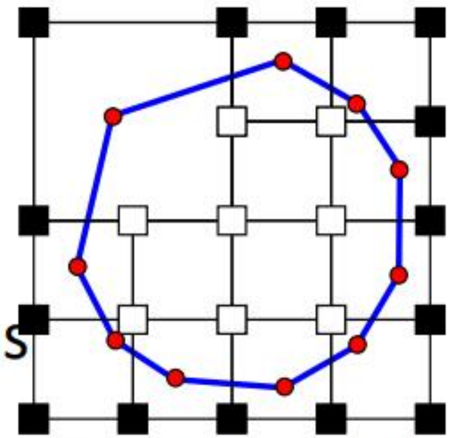


Dual Contouring

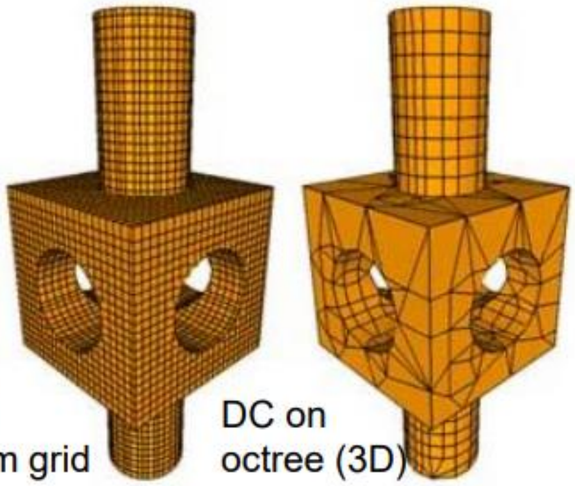
Slide Credit: Tao Ju

Primal vs. Dual

- Marching Cubes
  - ✓ Always manifold
  - ✗ Requires look-up table in 3D
  - ✗ Often generates thin and tiny polygons
  - ✗ Restricted to uniform grids
- Dual Contouring
  - ✗ Can be non-manifold
  - ✓ No look-up table needed
  - ✓ Generates better-shaped polygon
  - ✓ Can be applied to any type of gric



DC on a Quadtree (2D)



DC on uniform grid

DC on octree (3D)