Formal Analysis of Fractional Order Systems in HOL

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***Abstract*—Fractional order systems, which involve integration and differentiation of non integer order, are increasingly being used in the fields of control systems, robotics, signal processing and circuit theory. Traditionally, the analysis of fractional order systems has been performed using paper-and-pencil based proofs or computer algebra systems. These analysis techniques compromise the accuracy of their results and thus are not recommended to be used for safety-critical fractional order systems. To overcome this limitation, we propose to leverage upon the high expressiveness of higher-order logic to formalize the theory of fractional calculus, which is the foremost mathematical concept in analyzing fractional order systems. This paper provides a higher-order-logic formalization of fractional calculus based on the Riemann-Liouville approach using the HOL theorem prover. To demonstrate the usefulness of the reported formalization, we utilize it to formally analyze some fractional order systems, namely, a fractional electrical component Resistoductance, a fractional integrator and a fractional differentiator circuit.**

I. INTRODUCTION

In reality, many situations arise when integer order calculus is not sufficient to model all kind of dynamics. For example, an electrical component Resistoductance [10] exhibits an intermediate behavior between that of a resistor and inductor and thus its accurate modeling involves the differentiation of order between 0 and 1. Such systems that involve integration and differentiation of non integer order, or *fractional calculus* [26], for their modeling are usually referred to as fractional order systems. The idea of fractional calculus is as old as integer order calculus itself. The question which gave birth to fractional calculus was about the interpretation of $\frac{d^n y}{dx^n}$, if n is not an integer or more broadly if n is any real, irrational or even a complex number.

Accurate modeling of engineering and scientific systems have become imperative these days due to their extensive usage in safety-critical domains, such as, medicine and transportation. This fact has led to the widespread usage of fractional calculus in modeling physical systems. For example, in control engineering the concept of fractional operations is mostly used in fractional system identification [17], biomimetic control [6], fractional PI^{α} [22] and PD^{μ} controllers [8]. In signal processing, fractional operators are used in the design of fractional order differentiators and integrators [21] and for modeling the speech signals [20]. Other interesting applications of fractional calculus are in image processing [29], electromagnetic theory [13], chaotic communication [1], and circuit theory [10].

Traditionally, the analysis of fractional calculus based models has been done using paper-and-pencil proof methods. However, considering the complexity of present age engineering and scientific systems, such analysis is notoriously difficult if not impossible, and is quite error prone. Many examples of erroneous paper-and-pencil based proofs are available in the open literature, a recent one can be found in [7] and its identification and correction is reported in [27]. One of the most commonly used computer based analysis technique for fractional order systems is numerical computation of fractional integration and differentiation. Some examples include, chaos in fractional order volta systems [30], fractional PI^{α} controllers [22] and motion planning of redundant and hyper-redundant manipulators [23]. Fractional order systems are continuous in nature and thus the first step in their simulation based analysis is to construct a discretized system model with minimal error. Most of the numerical algorithms are based either on the Grünwald-Letnikov definition [12] or on the Power Series Expansion (PSE) method [30]. Both of them cannot provide reliable results due to the involvement of infinite summations in case of Grünwald-Letnikov definition and huge memory requirements in case of the PSE method. Similarly, the computation of the Gamma function $\Gamma(x)$ for large values of x is not possible in such numerical computation software packages. For example, MATLAB [24] returns `7.26e306` as the approximated value computed for $x = 171$ and returns `Inf` for all values beyond $x = 171$. Another alternative to analyze fractional order systems is computer algebra systems [3], which are very efficient for computing mathematical solutions symbolically, but they are not reliable [15] due to their limitations of dealing with side conditions. Another limitation of computer algebra systems related to fractional calculus is the uncertain simplification of singular expressions particularly in case of the Gamma function, which are frequently used in fractional calculus [18] . Another source of inaccuracy in computer algebra systems is the presence of unverified huge symbolic manipulation algorithms in their core, which are quite likely to contain bugs. Thus, these traditional techniques should not be relied upon for the analysis of fractional order systems, especially when they are used in safety-critical areas (e.g., cardiac tissue electrode interface [9] which is modeled and analyzed using fractional calculus), where inaccuracies in the analysis may even result in the loss of human lives.

In the past couple of decades, formal methods have been successfully used for the precise analysis of a variety of hardware and software systems. The rigorous exercise of developing a mathematical model for the given system and analyzing this model using mathematical reasoning usually increases the chances for catching subtle but critical design errors that are often ignored by traditional techniques like numerical methods. Given the sophistication of the present age fractional order systems and their extensive usage in safety critical applications, there is a dire need of using formal methods in this domain. However, due to the continuous nature of the analysis and the involvement of transcendental functions, automatic state-based approaches, like model checking [19], cannot be used in this domain. On the other hand, we believe that higher-order-logic theorem proving [14] offers a promising solution for conducting formal analysis of fractional order systems. The main reason is being the highly expressive nature of higher-order logic, which can be leveraged upon to essentially model any system that can be expressed in a closed mathematical form. In fact, most of the classical mathematical theories behind elementary calculus, such as limits, differentiation, integration and transcendental functions, have been formalized in higher-order logic [15]. In this paper, we build upon the available theories of elementary calculus to formalize Riemann-liouville's [26] definitions of fractional integration and differentiation in higher-order logic. These definitions are then used to formally verify some classical properties of fractional calculus using the HOL theorem prover [34], which has been chosen due to the availability of Harrison's seminal work on the formalization of elementary calculus [15]. The formal verification of these classical properties of fractional calculus , such as, linearity, identity and the relationship with elementary calculus, not only ensures the correctness of our formal definitions of fractional integration and differentiation but also plays a vital role in the formal analysis of fractional order systems. To the best of our knowledge, the reported formalization is the first one of its kind and facilitates the formal analysis of fractional order systems, which is a novelty that has not been presented in the open literature so far using any formal technique.

The rest of the paper is organized as follows: Section II describes some fundamentals of fractional calculus , its commonly used definitions and the justification behind the choice of Riemann-Liouville approach for our formalization. Section III presents the proposed framework for the formal analysis of fractional order systems. Section IV presents our HOL formalization. In order to demonstrate the practical effectiveness and the utilization of proposed framework, we present the analysis of some real-world fractional order systems, i.e., a Resistoductance, a fractional differentiator and a fractional integrator circuit in Section V. Finally, Section VII concludes the paper.

II. Fractional Calculus

In 1695, L'Hôpital asked Leibnitz regarding his notation $\frac{d^n y}{dx^n}$, "*What if n is* $\frac{1}{2}$". Leibnitz prophesied in his letter [10] to L'Hôpital,"*. . . Thus it follows that* $d^{\frac{1}{2}}x$ *will be equal to* $x\sqrt{dx:x}$. *This is an apparent paradox from which, one day, useful consequences can be drawn . . .*". Leibnitz's initial work on the problem of defining the derivative of arbitrary order gave birth to a new field of research in mathematics and attracted attention of many biologists, physicists, engineers and geometers. Initially more efforts were made for defining fractional derivatives and fractional integrals but Neils Henrik Abel [26] was the first one to use this idea in solving the famous Tautochrone problem. The other great mathematicians and physicists who touched the field of fractional calculus are Riemann, Liouville, Laurent, Heaviside, Al-Bassam, Davis Erdelyi, Riesz and Thomas J. Osler [26].

Fractional integrals and fractional derivatives are also referred to as Differintegrals [28] and there are more than ten well known definitions for Differintegrals [9]. We consider two of them, which are most widely used in analyzing real-world problems. These are the Riemann-Liouville and Grünwald-Letnikov definitions, which are also equivalent for a wide class of functions [31].

- Riemann-Liouville (RL) Definition:

$$J_a^v f(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t) dt \tag{1}$$

Where $J_a^v f(x)$ represents fractional integration with order v and lower integration limit a. $a = 0$ gives the Riemann definition and $a = -\infty$ gives the Liouville definition of fractional integration [32]. Γ in the above definition denotes the Gamma function which is defined using the well-known improper integral as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{2}$$

for $z > 0$.
The fractional differentiation is given as follows:

$$D^v f(x) = (\frac{d}{dx})^m J_a^{m-v} f(x) \tag{3}$$

where m represents the ceiling of v, i.e., $\lceil v \rceil$.

- Grünwald-Letnikov (GL) Definition:

$${}_cD_x^v f(x) = \lim_{h \to 0} h^{-v} \sum_{k=0}^{[\frac{x-c}{h}]} (-1)^k \binom{v}{k} f(x-kh) \tag{4}$$

Grünwald-Letnikov definition caters for both fractional differentiation and integration, as positive values of v give fractional differentiation and negative values of v give fractional integration. Here, $\binom{v}{k}$ represents the binomial coefficient, which is described in terms of the Gamma function.

The Riemann-Liouville definition provides a way to find analytical solutions while Grünwald-Letnikov definition facilitates the numerical computation of solutions. There are two motivations of using the Riemann-Liouville definition for our formalization: Firstly, it is widely used in the modeling and analysis of engineering fractional order systems [10], Secondly, the analysis carried out in this way is purely analytical and hence free from any kind of approximations. On the other hand, Grünwald-Letnikov definition is more suitable for numerical analysis based methods and thus provides approximate solutions.

III. Proposed framework

The proposed framework, given in Figure 1, outlines the main idea behind the theorem-proving-based fractional order system analysis. The grey shaded boxes in this figure represent the key contributions of the paper that serves as the fundamental requirements of conducting fractional order system analysis in a theorem prover. Like all the system analysis tools, the input to this framework, depicted by two rectangles with curved bottoms, is the description of the fractional order system that needs to be analyzed and a set of properties that are required to be checked for the given system.

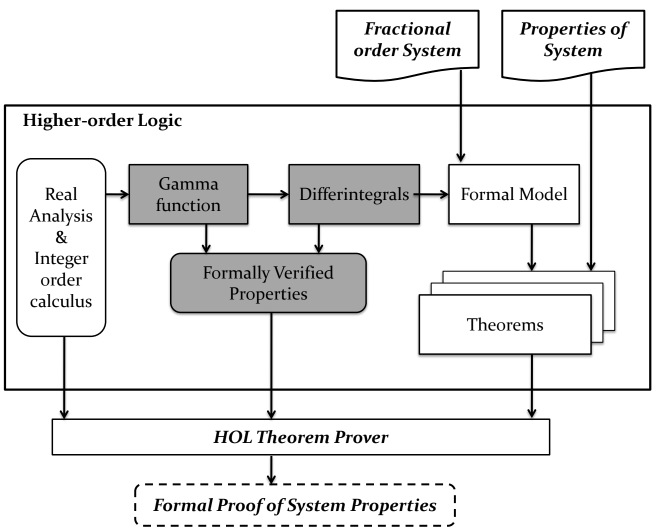

Fig. 1. Proposed Framework

The first step in conducting fractional order system analysis using a theorem prover is to construct a formal model of the given system in higher-order logic. For this purpose, the foremost requirement is the ability to formalize fractional derivatives and integrals (Differintegrals) as higher-order logic functions. The formalization of Differintegrals, given in Equations 1 and 3, requires the mathematical theories of real numbers, integer order calculus and the Gamma function. Harrison's work on the formalization of real numbers [15] provides the first two requirements and we built upon Harrison's work to formalize the Gamma function in this paper to fulfil the third requirement. Using these fundamentals, this paper also presents the formalization of Differintegrals, given in Equation 1 and 3, which in turn can be used to represent the dynamics of fractional order systems in higher-order logic. The second step in the theorem proving based fractional order system analysis is to utilize the formal model of fractional order system, developed in the first step, to express system properties as higher-order logic theorems.

The third step for conducting fractional order system analysis in a theorem prover is to formally verify the higher-order-logic theorems developed in the previous step using a theorem prover. For this verification, it would be quite handy to have access to a library of some pre-verified theorems corresponding to some commonly used properties of Gamma function and Differintegrals. To fulfil this requirement, this paper presents formal verification of the classical properties of Gamma function, such as, Pseudo-Recurrence Relation, Factorial Generalization and Functional Equation, and Differintegrals, such as, Identity and Linearity, using the HOL theorem prover. Building on such a library of theorems would minimize the interactive verification efforts and thus speed up the verification process. Finally, the output of the theorem proving based fractional order system analysis framework, depicted by the rectangle with dashed edges, is the formal proofs of system properties that certify that the given system properties are valid for the given fractional order system.

IV. HOL Formalization

This section presents the higher-order logic formalization of the main requirements of the proposed framework, depicted by the gray shaded boxes in Figure 1. We have arranged the information in two subsections. The first subsection presents the formalization of the Gamma function and the formal verification of its associated properties using the HOL theorem prover. While the second subsection presents formalization of Riemann-Liouville definition of Differintegrals and the formal verification of its associated properties in HOL.

A. Formalization of Gamma function

The applicability of Gamma function in fractional calculus is due to its unique characteristic of generalizing factorials over non-integer numbers. The theory of improper integrals [2] suggests that it is convenient to write Gamma function (Equation 2) as follows:

$$\Gamma(z) = \lim_{a \to 0_+, b \to \infty} \int_a^b t^{z-1} e^{-t} dt \tag{5}$$

We formalize Equation (5) as follows:

Definition 1: *Gamma Function*

$\vdash \forall$ z.gamma z =
  lim(λ n.(lim
    (λ b.$\int_{\frac{1}{2^n}}^{b}$ t rpow (z-1)exp(-t) dt))

The function `rpow` [33] is a power function with real exponent. It takes two real numbers `x` and `y`, and returns x^y. We used $\lim_{n\to\infty}(\frac{1}{2^n})$ to model 0_+ as $(\frac{1}{2^n})$ becomes very close to 0 as n becomes very large. The integral $(\int_a^b f)$ is used to represent HOL function `integral(a,b) f`, which represents the formalization of the Gauge integral in HOL [15]. Mhamdi [25] presented the higher-order logic formalization of Lebesgue integration theory, which is fundamental concept in many

TABLE I
PROPERTIES OF THE GAMMA FUNCTION

Property	HOL Formalization
Pseudo-Recurrence Relation	⊢ ∀ z.(0 < z) ⟹ (gamma (z + 1)= z gamma (z))
Functional Equation	⊢ gamma 1 = 1
Factorial Generalization	⊢ ∀ n ∈ℕ. gamma(n + 1) = n!
Reconstruction of Gamma	⊢ ∀ x z.(0 < z)∧(0 < x) ⟹ gamma z = gamma_upper x z + gamma_lower x z)
Recurrence Lower_Gamma	⊢ ∀ z x.(0 < z)∧(0 < x) ⟹ gamma_lower x (z + 1)= (z)gamma_lower x z − x rpow (z)exp(-x)
Recurrence Upper_Gamma	⊢ ∀ z x.(0 < z)∧(0 < s) ⟹ gamma_upper s z = (z − 1)gamma_upper s (z-1)+ s rpow (z-1)exp(-s)

mathematical theories and allows a wider class of functions than the Riemann integration theory. In our formalization, we built upon Harrison's formalization of Gauge integral because the proposed framework is intended to be used by engineers and practitioners, who are normally not familiar with Lebesgue integration theory.

The lower and upper incomplete Gamma functions play a vital role in obtaining Differintegrals of periodic functions, such as, sinusoidal response study of fractional operators [10], and can be formalized as follows:

Definition 2: *Upper Incomplete Gamma Function*

$$\vdash \forall\ \texttt{x s.gamma\_upper s z} = (\texttt{lim}\ (\lambda\ \texttt{b}.\ \int_{s}^{b} \texttt{t rpow (z-1)exp(-t) dt}))$$

Definition 3: *Lower Incomplete Gamma Function*

$$\vdash \forall\ \texttt{x z.gamma\_lower x z} = (\texttt{lim}\ (\lambda\ \texttt{n}.\ \int_{\frac{1}{2^n}}^{x} \texttt{t rpow (z-1)exp(-t) dt}))$$

Next, we defined and verified some of the key properties of Gamma function in HOL using Definitions 1, 2, and 3. The formal verification of these properties not only ensures the correctness of our formal definition but also facilitates the formal reasoning about fractional order systems in higher-order logic as mentioned in Section III. The formally verified properties of Gamma function are given in Table I.

The first property in Table I represents the Pseudo-Recurrence Relation of the Gamma function and can be classified as the most important property of the Gamma function as it plays a vital role in verifying the other properties of Gamma function and Differintegrals. The verification of this property was also one of the most challenging part of our formalization as it involves the core concepts of improper integrals, limits and sequences. Its reasoning process involve ten main lemmas, such as, convergence of integral with respect to upper and lower limits, limits on infinity and zero, simplification of integrand by integration by parts and the continuity, differentiability and integrability of the integrand. The complete formalization details are provided in [33].

The second and third properties of Table I, i.e, Functional Equation and Factorial Generalization, are very important in establishing the link between fractional calculus and integer order calculus. The verification of these properties requires the Pseudo-Recurrence Relation of Gamma function along with some limit theory proofs and arithmetic reasoning in HOL. The fourth property, Reconstruction of Gamma function, shows that the Gamma function can be divided into two integrals, which are incomplete at one limit, i.e., upper and lower incomplete Gamma functions. The verification of this property requires lemmas used in the verification of Pseudo-Recurrence Relation along with the properties of the Gauge integral. The last two properties in Table I show the recurrence relation of the upper and lower incomplete Gamma functions. These relations are very important in fractional calculus because factional integration and differentiation of many important functions, e.g., Exponential function, is represented in terms of the incomplete Gamma functions and then these properties are utilized to evaluate such mathematical expressions. The verification of these properties is similar to that of the verification of Pseudo-Recurrence Relation.

This completes our formalization of the Gamma function, which to the best of our knowledge is the first one in higher-order logic. The main challenge in the reasoning process is to deal with improper integrals in higher-order logic. The Gamma function is useful in many domains, such as, probability theory (Gamma Distribution), and our formalization can be directly utilized in such applications. Our formalization of Gamma function can be generalized to formalize other improper integrals, such as, the Beta function. Next, we build upon the formalization of the Gamma function to formalize Differintegrals.

B. Formalization of Differintegrals

The second major requirement of formal reasoning about fractional order systems, is the formalization of Differintegrals, as depicted in Figure 1. We utilize Equations (1) and (3) to formally define fractional integration and differentiation, respectively.

Definition 4: *Fractional Integration*

$$\vdash \forall\ \texttt{f v a x.frac\_int f v a x} = \texttt{if (v = 0) then f else } \texttt{lim}(\lambda \texttt{n}.\ \frac{1}{\texttt{gamma v}} (\int_{a}^{x-\frac{1}{2^n}} \texttt{((x - t) rpow (v-1)) f(t) dt})$$

Definition 5: *Fractional Differentiation*

$$\vdash \forall\ \texttt{f v a x. frac\_diff f v a x} = \texttt{n\_order\_deriv (clg v) (frac\_int f (clg v - v) a x)}$$

Where `f` is a function of type (`real` → `real`), `v` is a `real` number that indicates the order of integration/differentitiation, and `a` and `x` represent the lower and upper limits of integration, respectively. The function `n_order_deriv` returns the n^{th} integer order derivative of its argument `f` as $\frac{d^n f}{dx^n}$. The function `clg` is the ceiling function, which returns the least greater

TABLE II
PROPERTIES OF DIFFERINTEGRALS

Property	HOL Formalization
Identity	⊢ ∀ f a x. (a < x) ⟹ (frac_int f 0 a x = f)∧ (frac_diff f 0 a x = f)
Generalized Integral	⊢ ∀ f a x v ∈ N. (a < x)∧ (1 < v) ⟹ frac_int f v a x = lim(λn. $\frac{1}{(v-1)!} \int_a^{x-\frac{1}{2^n}}$ (x - t) rpow (v-1)f(t) dt)
frac_int Linearity	⊢ ∀ f v x a b. (frac_exists f x v)∧ (frac_exists g x v) ⟹ frac_int (a f + b g) v 0 x = a(frac_int f v 0 x)+ b(frac_int g v 0 x)
frac_diff Linearity	⊢ ∀ f v x a b. (frac_exists f x v)∧ (frac_exists g x v)∧ (∀ m. (m <= clg v) ⇒ (n_order_deriv m (frac_int f v 0 x)) differentiable x)∧ (∀ m. (m <= clg v) ⇒ (n_order_deriv m (frac_int g v 0 x)) differentiable x)⟹ (frac_diff (a f + b g) v 0 x = a(frac_diff f v 0 x)+ b(frac_diff g v 0 x))

integer of its real number argument. It is important to note that we have explicitly defined the case for $\mathtt{v} = 0$, which is justified based on integer order calculus and proves to be very convenient for further manipulations [11].

As mentioned in Section III, now we will use our formal definitions of Differintegrals to formally verify some of the classical properties of fractional calculus, given in Table II, using the HOL theorem prover. The first property of Differintegrals is the Identity property, which shows that the 0^{th} order fractional operators return original functions. The proof of the first part of this property is obvious from the definition of fractional integration (Definition 4) and proof of the second part is done based on the fact that $\frac{d^n f}{dx^n}$ with order 0 returns the original function. The second property in Table II shows that fractional integration generalizes the integer order integration. The verification of this property utilizes the third property (Factorial Generalization) of Gamma function, given in Table I. The next property is about the linearity of fractional integration and helps in formal reasoning about fractional order systems with multiple inputs. In the HOL formalization of `frac_int linearity` property, the assumptions `frac_exists f x v` and `frac_exists g x v` ensure the existence of Differintegrals for function `f` and `g`, respectively. The verification of this property requires the properties of Gamma function, integer order integration and limits along with some arithmetic reasoning. The HOL formalization of last property of Differintegrals in Table II shows the linearity of fractional differentiation. From Definition 5 it is clear that fractional differentiation involves fractional integration followed by the n^{th} order ordinary differentiation. So, the third and fourth assumptions of this property ensures the differentiability of $(clg(v)-v)^{th}$ order fractional integral of the functions $f(t)$ and $g(t)$, respectively. The formal verification of this property requires the linearity of the n^{th} integer order derivative along with some arithmetic reasoning.

Due to inherent soundness of higher-order logic theorem proving, our verification results are exactly the same as produced by paper-and-pencil proof methods. It is interesting to note that we have been able to identify a couple of critical assumptions that are missed by almost all the paper-and-pencil based proof analysis, that we came across. For example, the assumption $\mathtt{0} < \mathtt{x}$ in the last two properties of the Gamma function (Table I) have not been specified in anyone of the paper-and-pencil proof based analysis (e.g., [10]). Obviously the results do not hold without this assumption and this discrepancy in the paper-and-pencil based proofs may lead to disastrous consequences if these properties are used without considering $\mathtt{0} < \mathtt{x}$ for designing safety-critical fractional order systems.

The formalization, presented in this section, had to be done in an interactive way due to the undecidable nature of higher-order logic and took around 7000 lines of HOL code and approximately 550 man hours. However, the main advantage of this rigorous exercise is that our results can be built upon to facilitate formal reasoning about fractional order systems. Our proof script is available for download [33] and thus can be utilized by other researchers to conduct the formal analysis of their fractional order systems.

V. APPLICATIONS

In order to illustrate the utilization and effectiveness of the proposed framework, we apply it to analyze three real-world fractional order systems, i.e., a fractional electrical component Resistoductance, a fractional integrator and a differentiator circuit. Resistoductance is used to extend the current-voltage relationship to non-integer order and this kind of fractional order model is usually used for modeling bio-electrodes for cardiac tissue interfacing [9]. Fractional integrators and differentiators are the most basic components in fractional order PID (proportional integrator differentiator) controllers and can achieve more robustness than integer order control [5]. These systems have been chosen as case studies in our work because of their wide usability in the field of circuit theory and control systems. To the best of our knowledge, currently, there is no formal technique available for the formal verification of such systems.

A. Resistoductance

Electrical components, such as, resistors, inductors and capacitors are largely used to perform integer order calculus operations for different engineering and scientific applications. However, actual electrical components do not posses ideal behavior and exhibit some fractional order characteristics. Ignoring these characteristics always results in modeling inaccuracies. Therefore, fractional calculus is being widely used to capture real world dynamics of electrical components these days [4]. Resistoductance is a linear electrical circuit element that posses the characteristics between an ohmic resistor and an inductor. Being a fractional order electrical component, it exhibits fractional order dynamics, which can be modeled by Differintegrals. The model of a single Resistoductance is shown in Figure 2, and its governing voltage and current relationship is given as follows:

$$i(t) = \frac{1}{K} J^{\alpha} v(t) \tag{6}$$

where $v(t)$ is the voltage and $i(t)$ is the current through the circuit element at time t. The range of the α is between 0 and 1. If $\alpha = 0$ the circuit will be purely resistive with K = R ohms and if $\alpha = 1$ the circuit will be purely inductive with K = L henrys.

The two important characteristics of Resistoductance are the output current through the circuit element when constant input voltage V_0 is applied and the behavior of the output current for the cases when $\alpha = 0$ and $\alpha = 1$. These two properties are widely used in designing Resistoductance based fractional order systems for signal processing and control engineering applications [4].

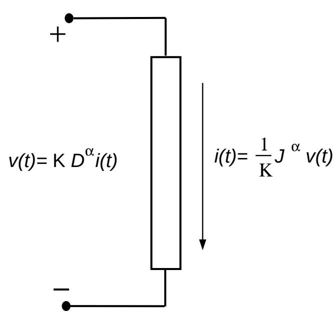

Fig. 2. Resistoductance

Now, we will present the formal verification of the above mentioned two properties of Resistoductance using our proposed framework given in Figure 1. The first step in conducting the formal analysis of Resistoductance is to construct its formal model in higher-order logic. Due to the availability of Definition 4, the formalization can be simply done as follows:

Definition 6: *Current through Resistoductance*

```
⊢ ∀ K v_i alpha x. i_t K v_i alpha x =
          (1/K)frac_int v_i(t) alpha 0 x
```

where `v_i` is input voltage, `i_t` is current through the circuit element, `alpha` is the order of integration, and the variable `x` represents the upper limit of integration. In the above definition the lower limit of integration is taken as `0` [10]. The next step, in the proposed framework, is to utilize the formal model of Resistoductance (Definition 6) to express the properties of interest as higher-order logic theorems as follows:

Theorem 1: *i_t for constant voltage V_0*

```
⊢ ∀ K v_0 alpha x.
     (0 < x) ∧ (0 < alpha) ⟹
(i_t K V_0 alpha x =
     (1/K (Gamma (alpha + 1))
                    (V_0(x rpow alpha))))
```

Theorem 2: *Special Cases for i_t*

```
⊢ ∀ x. (0 < x) ⟹
  (((alpha = 0) ⇒
      (i_t K V_0 alpha x = V_0 / K)) ∧
   ((alpha = 1)⇒
   (i_t K V_0 alpha x = (V_0 / K ) x)))
```

Theorem 1 shows the relationship of output current of Resistoductance when constant input voltage `V_0` is applied at $t = 0$. The formal verification of this theorem is based on the properties of Gamma function (Table I, Pseudo-Recurrence relation) and the definition of fractional integration. Since, these required properties have already been verified in HOL library, the interactive formal reasoning process only consists of verifying the continuity of fractional integral. Theorem 2 shows an interesting feature of Resistoductance, i.e., for ($\texttt{alpha} = \texttt{0}$) it behaves as a pure resistor and for ($\texttt{alpha} = \texttt{1}$) it exhibits the behavior of a pure inductor. The verification of Theorem 2 requires Theorem 1, the properties of the real power (`rpow`) function and some arithmetic reasoning.

This verification of Theorems 1 and 2 consumed approximately 350 lines of HOL code and about two man hours and thus was very short compared to the challenging verification of the theorems presented in the last section. The verification process, besides being compact, was also very straightforward and involved reasoning based on real analysis theories only and thus can be done with some basic know how of higher-order-logic theorem proving. The main reason for the above mentioned benefits is clearly the availability of formalized Gamma function and the Differintegrals.

B. Fractional Differentiator and Integrator Circuits

Proportional integrator (PI) and proportional integrator differentiator (PID) controllers are widely used in the industry. Numerous reliable and high performance controllers have been designed and deployed. In recent years, it has been observed that Fractional order (FO) controllers offer more flexibility in the adjustment of gain and phase characteristics than integer order controllers. Due to these flexibilities, there is a growing interest in using fractional order controllers in industry and academia [5]. The most fundamental components of PI and PID controllers are integrator and differentiator circuits, respectively. In this section, we will present the formal analysis of a fractional integrator and a differentiator circuit, [10] shown in Figure 3. The output voltage-current equations for a fractional integrator and a differentiator circuits are given as follows:

$$v_o(t) = -\frac{1}{RC} J^{\mu} v_i(t) \qquad Integrator \tag{7}$$

$$v_0(t) = -RCD^{\mu} v_i(t) \qquad Differentiator \tag{8}$$

where R and C denotes resistance and capacitance, respectively, and their values are used to define the reset rate of PID controllers. The variables, $v_o(t)$ and $v_i(t)$, in the

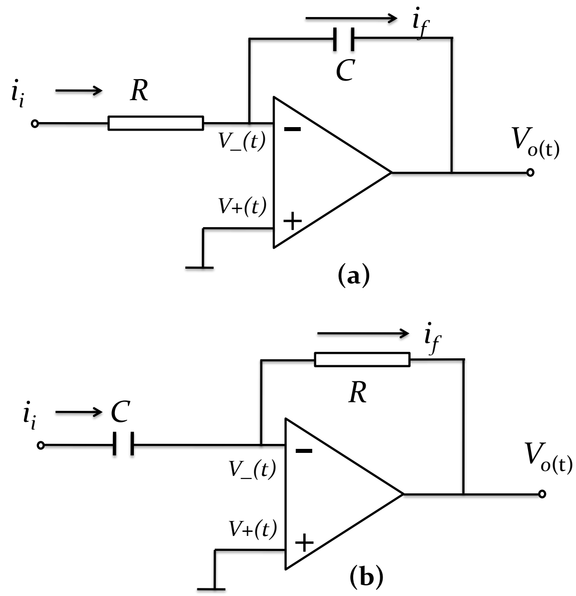

Fig. 3. (a) Integrator (b) Differentiator

above Equations represent output and input voltages at time t, respectively.

The output response of integrator and differentiator circuit is usually analyzed for benchmark input signals, such as, the unit step, which is defined as follows:

$$u(t) = \begin{cases} 0 & \text{if } t \leq 0; \\ 1 & \text{if } t > 1; \end{cases}$$

The first step in the formal analysis of integrator and differentiator circuits, when unit step signal is applied at the input, is to construct the formal model of these circuits and unit step signal in higher-order logic. Since the governing equations (Equations 7, 8) of integrator and differentiator circuits involve fractional integration and differentiation, thus we utilize our formalized definitions (Definition 4, 5) of Differintegrals as follows:

Definition 7: *Fractional Order Integrator*

```
⊢ ∀ R C v_i mu x. v_I_0 R C v_i mu x =
              -(1/RC)frac_int v_i(t) mu 0 x
```

Definition 8: *Fractional Order Differentiator*

```
⊢ ∀ R C v_i mu x. v_D_0 R C v_i mu x=
              -(RC)frac_diff v_i(t) mu 0 x
```

Definition 9: *Unit Step*

```
⊢ ∀ t. unit t = if (0 <= t) then 1 else 0
```

where v_I_0 and v_D_0 are output voltages of integrator and differentiator circuits, respectively. v_i is the input voltage, R, C, mu and x represent resistance, capacitance, order of integration/differention and upper limit of integration, respectively.

Now, the next step in the formal analysis of fractional integrator and differentiator, as mentioned in Fig 1, is to describe their properties of interest as higher-order logic theorems:

Theorem 3: *Output of Fractional Integrator Circuit*

```
⊢ ∀ R C mu x .
      (0 < x) ∧ (0 < mu) ∧ ( mu < 1)⟹
(v_I_0 R C (unit t) mu x =
      (-1/(RC Gamma (mu + 1))
                          ((x rpow ( mu))))
```

Theorem 4: *Output of Fractional Differentiator Circuit*

```
⊢ ∀ R C mu x .
      (0 < x) ∧ (0 < mu) ∧ ( mu < 1)⟹
(v_D_0 R C (unit t) mu x =
      ((-1/(RC (Gamma ((1 - mu))))
                         ((x rpow (- mu))))
```

The next step in the theorem proving based fractional order system analysis is the verification of above mentioned theorems using the already verified properties and lemmas of Section IV. Theorem 3 gives the output response of a fractional integrator circuit for unit step signal, and its formal verification certifies the output response under the given conditions. The availability of already verified properties of Gamma function and Differintegrals (Table I and Table II) led us to the simple subgoal, i.e., the proof of continuity of the integrand, which involves multiplication of power function and the unit step signal. We verified the continuity by differentiability using the classical definition of derivative formalized in HOL [15].

Theorem 4 describes the output response of the fractional differentiator circuit with unit step signal as an input. The second and third assumptions in Theorem 4 ensure that the order of the fractional differentiation mu is between 0 and 1 which means that clg of mu will always be 1. So the verification of this theorem requires fractional integration of the order 1 − mu followed by the fractional differentiation of order 1. This requires Theorem 3 along with some arithmetic reasoning. Just like the case of the Resistoductance, the verification of Theorem 3 and Theorem 4 was very straightforward and took about 400 lines of HOL code and about 2.5 man hours.

The above case studies clearly demonstrate the effectiveness of the proposed theorem proving based fractional order system analysis technique. Due to the formal nature of the model and inherent soundness of higher-order logic theorem proving, we have been able to verify the properties of given fractional order systems with 100% accuracy; a feature that, to the best of our knowledge, is not available in any other computer based analysis technique. This additional benefit comes at the cost of the time and effort spent, while formalizing the Differintegrals and formally reasoning about their properties. But, the availability of such a formalized infrastructure significantly reduces the time required to analyze fractional order systems, as the verification task of the properties of Resistoductance and a fractional integrator and differentiator circuits took just a couple of man hours each.

VI. Conclusions

In this paper, we presented a novel application of formal methods in the area of analyzing fractional order systems. In particular, we developed a framework for accurate and reliable analysis of fractional order systems within the sound core of the HOL theorem prover. This approach can thus be of great benefit for the analysis of fractional order systems used in safety-critical domains, such as, medicine and transportation. The paper provides the complete formalization details of Differintegrals along with the formal verification of their classical properties. For illustration purposes, we provided the formal analysis of Resistoductance, a fractional differentiator and a fractional integrator circuit. To the best of our knowledge, this is the first time that a formal method technique has been used to conduct the analysis of fractional order systems.

The reported formalization opens the doors to many interesting and novel directions of research. Some worth mentioning ones include enriching the library of the formally verified properties of Differentagrals with law of exponents and relationship with Beta function to broaden the scope of formal fractional order system analysis. Similarly, the reported formalization can be utilized to formalize the fractional Laplace transform theory, which in turn can be utilized for the the formal analysis of industrial fractional order control systems. Our formalization was done using real numbers and the same formalization can also be extended to cover the complex numbers using the higher-order-logic formalization of complex number theory [16], which would allow us to formalize fractional electromagnetic systems, such as, fractional rectangular waveguides [13].

VII. Acknowledgement

This work was supported by the National Research Program for Universities grant (number 1543) of Higher Education Commission (HEC), Pakistan.

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