CS243: Discrete Structures

First Order Logic, Rules of Inference

Işıl Dillig

Announcements

- Homework 1 is due now
- Homework 2 is handed out today
- Homework 2 is due next Tuesday

Review of Last Lecture

- Building blocks in FOL: constants, variables, predicates
- Formulas formed using predicates, connectives, and quantifiers
- Truth value of FOL formulas depend on universe of discourse and interpretation of predicates and variables
- Universal quantification $\forall x.P(x)$ is true if P is true for all objects in universe of discourse
- Existential quantification $\exists x.P(x)$ is true if there exists an object for which P is true

Translating English into First-Order Logic

Given predicates $student(x)$, $atWM(x)$, and $friends(x, y)$, how do we express the following in first-order logic?

- "Every William&Mary student has a friend"
- "At least one W&M student has no friends"
- "All W&M students are friends with each other"

Satisfiability, Validity in FOL

- The concepts of satisfiability, validity also important in FOL
- An FOL formula F is satisfiable if there exists some domain and some interpretation such that F evaluates to true
- Example: Prove that $\forall x.P(x) \rightarrow Q(x)$ is satisfiable.
- An FOL formula F is valid if, for all domains and all interpretations, F evaluates to true
- Prove that $\forall x.P(x) \rightarrow Q(x)$ is not valid.
- Formulas that are satisfiable, but not valid are contingent, e.g., $\forall x.P(x) \rightarrow Q(x)$

Equivalence

- Two formulas F_1 and F_2 are equivalent if $F_1 \leftrightarrow F_2$ is valid
- In PL, we could prove equivalence using truth tables, but not possible in FOL
- However, we can still use known equivalences to rewrite one formula as the other
- Example: Prove that $\neg(\forall x.\ (P(x) \rightarrow Q(x)))$ and $\exists x.\ (P(x) \wedge \neg Q(x))$ are equivalent.
- Example: Prove that $\neg\exists x.\forall y.P(x, y)$ and $\forall x.\exists y.\neg P(x, y)$ are equivalent.

Motivation for Proof Rules

- Learned how to express various facts in logic, but this is not all that useful on its own
- The reason logic is useful: allows formalizing arguments, constructing validity proofs, and make inferences
- Rest of lecture: learn about proof rules for logic
- By applying proof rules, can make logical inferences that are correct by construction

Rules of Inference

- Proof rules are written as rules of inference:

$$\frac{\begin{array}{c}\text{Hypothesis1}\\ \text{Hypothesis2}\\ \ldots\end{array}}{\text{Conclusion}}$$

- An example inference rule:

$$\frac{\begin{array}{c}\text{All men are mortal}\\ \text{Socrates is a man}\end{array}}{\therefore \text{Socrates is mortal}}$$

- Valid inference rule, but too specific
- We'll learn about more general inference rules that will allow constructing formal proofs

Modus Ponens

- Most basic inference rule is modus ponens:

$$\frac{\begin{array}{l}\phi_1\\ \phi_1 \rightarrow \phi_2\end{array}}{\phi_2}$$

- This rule is valid because we know ϕ_1 is true, and by definition of implication, if ϕ_1 is true, then ϕ_2 must be true
- Modus ponens applicable to both propositional logic and first-order logic

Example Uses of Modus Ponens

- Application of modus ponens in propositional logic:

$$\frac{\begin{array}{l}p \wedge q\\ (p \wedge q) \rightarrow r\end{array}}{\quad}$$

- Application of modus ponens in first-order logic:

$$\frac{\begin{array}{l}P(a)\\ P(a) \rightarrow Q(b)\end{array}}{\quad}$$

Modus Tollens

- Second imporant inference rule is modus tollens:

$$\frac{\begin{array}{l}\phi_1 \rightarrow \phi_2\\ \neg\phi_2\end{array}}{\neg\phi_1}$$

- Recall: $\phi_1 \rightarrow \phi_2$ and its contrapositive $\neg\phi_2 \rightarrow \neg\phi_1$ are equivalent to each other
- Therefore, correctness of this rule follows from modus ponens and equivalence of a formula and its contrapositive.

Example Uses of Modus Tollens

- Application of modus tollens in propositional logic:

$$\frac{\begin{array}{l}p \rightarrow (q \vee r)\\ \neg(q \vee r)\end{array}}{\quad}$$

- Application of modus tollens in first-order logic:

$$\frac{\begin{array}{l}Q(a)\\ \neg P(a) \rightarrow \neg Q(a)\end{array}}{\quad}$$

Hypothetical Syllogism (HS)

$$\frac{\begin{array}{c}\phi_1 \rightarrow \phi_2 \\ \phi_2 \rightarrow \phi_3\end{array}}{\phi_1 \rightarrow \phi_3}$$

- Basically says "implication is transitive"
- Example:

$$\frac{\begin{array}{c}P(a) \rightarrow Q(b) \\ Q(b) \rightarrow R(c)\end{array}}{\quad}$$

Or Introduction

$$\frac{\phi_1}{\phi_1 \vee \phi_2}$$

- Correctness follows from definition of $\vee$
  - $\phi_1 \vee \phi_2$ is true if either ϕ_1 or ϕ_2 is true.
- Example application: "Socrates is a man. Therefore, either Socrates is a man or there are red elephants on the moon."
- The book calls this rule addition – feel free to use whichever term is more natural for you

Or Elimination

$$\frac{\begin{array}{c}\phi_1 \vee \phi_2 \\ \neg\phi_2\end{array}}{\phi_1}$$

- Either ϕ_1 or ϕ_2 is true. We know ϕ_2 is false. Therefore, ϕ_1 must be true.
- Example application: "It is either a dog or a cat. It is not a dog. Therefore, it must be a cat."
- The book calls this rule disjunctive syllogism; I call it Or Elimination – use whichever you prefer

And Introduction

$$\frac{\begin{array}{c}\phi_1 \\ \phi_2\end{array}}{\phi_1 \wedge \phi_2}$$

- This rule just follows from definition of conjunction
- Example application: "It is Tuesday. It's the afternoon. Therefore, it's Tuesday afternoon".
- The book calls this rule conjunction; I call it And Intro – use whichever you prefer

And Elimination

$$\frac{\phi_1 \wedge \phi_2}{\phi_1}$$

- This rule also just follows from definition of conjunction
- Example application: "It is Tuesday afternoon. Therefore, it is Tuesday".
- The book calls this rule simplification; I call it And Elimination – use whichever you prefer

Resolution

- Final inference rule: resolution

$$\frac{\begin{array}{c}\phi_1 \vee \phi_2 \\ \neg\phi_1 \vee \phi_3\end{array}}{\phi_2 \vee \phi_3}$$

- To see why this is correct, observe ϕ_1 is either true or false.
- Suppose ϕ_1 is true. Then, $\neg\phi_1$ is false. Therefore, by second hypothesis, ϕ_3 must be true.
- Suppose ϕ_1 is false. Then, by 1st hypothesis, ϕ_2 must be true.
- In any case, either ϕ_2 or ϕ_3 must be true; $\therefore$ $\phi_2 \vee \phi_3$

Resolution Example

- Example 1:

$$\frac{P(a) \vee \neg Q(b) \quad Q(b) \vee R(c)}{\quad}$$

- Example 2:

$$\frac{p \vee q \quad \neg p}{\quad}$$

Summary

Name	Rule of Inference
Modus ponens	ϕ_1, $\phi_1 \rightarrow \phi_2$ ⊢ ϕ_2
Modus tollens	$\phi_1 \rightarrow \phi_2$, $\neg\phi_2$ ⊢ $\neg\phi_1$
Hypothetical syllogism	$\phi_1 \rightarrow \phi_2$, $\phi_2 \rightarrow \phi_3$ ⊢ $\phi_1 \rightarrow \phi_3$
Or introduction	ϕ_1 ⊢ $\phi_1 \vee \phi_2$
Or elimination	$\phi_1 \vee \phi_2$, $\neg\phi_2$ ⊢ ϕ_1
And introduction	ϕ_1, ϕ_2 ⊢ $\phi_1 \wedge \phi_2$
And elimination	$\phi_1 \wedge \phi_2$ ⊢ ϕ_1
Resolution	$\phi_1 \vee \phi_2$, $\neg\phi_1 \vee \phi_3$ ⊢ $\phi_2 \vee \phi_3$

Using the Rules of Inference

Assume the following hypotheses:

1. It is not sunny today and it is colder than yesterday.
2. We will go to the lake only if it is sunny.
3. If we do not go to the lake, then we will go hiking.
4. If we go hiking, then we will be back by sunset.

Show these lead to the conclusion: "We will be back by sunset."

Encoding in Logic

- First, encode hypotheses and conclusion as logical formulas.
- To do this, identify propositions used in the argument:
  - s = "It is sunny today"
  - c= "It is colder than yesterday"
  - l = "We'll go to the lake"
  - h = "We'll go hiking"
  - b= "We'll be back by sunset"

Encoding in Logic, cont.

- "It's not sunny today and colder than yesterday."
- "We will go to the lake only if it is sunny"
- "If we do not go to the lake, then we will go hiking."
- "If we go hiking, then we will be back by sunset."
- Conclusion: "We'll be back by sunset"

Formal Proof Using Inference Rules

1.	$\neg s \wedge c$	Hypothesis
2.	$l \rightarrow s$	Hypothesis
3.	$\neg l \rightarrow h$	Hypothesis
4.	$h \rightarrow b$	Hypothesis

Another Example

Assume the following hypotheses:

1. It is not raining or Kate has her umbrella
2. Kate does not have her umbrella or she does not get wet
3. It is raining or Kate does not get wet
4. Kate is grumpy only if she is wet

Show these lead to the conclusion: "Kate is not grumpy."

Encoding in Logic

- First, encode hypotheses and conclusion as logical formulas.
- To do this, identify propositions used in the argument:
  - r = "It is raining"
  - u= "Kate has her umbrella"
  - w = "Kate is wet"
  - g = "Kate is grumpy"

Encoding in Logic, cont.

- "It is not raining or Kate has her umbrella."
- "Kate does not have her umbrella or she does not get wet"
- "It is raining or Kate does not get wet."
- " Kate is grumpy only if she is wet."
- Conclusion: "Kate is not grumpy."

Formal Proof Using Inference Rules

1.	$\neg r \lor u$	Hypothesis
2.	$\neg u \lor \neg w$	Hypothesis
3.	$r \lor \neg w$	Hypothesis
4.	$g \to w$	Hypothesis

Additional Inference Rules for Quantified Formulas

- Inference rules we learned so far are sufficient for reasoning about quantifier-free statements
- Four more inference rules for making deductions from quantified formulas
- These come in pairs for each quantifier (universal/existential)
- One is called generalization, the other one called instantiation

Universal Instantiation

- If we know something is true for all members of a group, we can conclude it is also true for a specific member of this group
- This idea is formally called universal instantiation:

$$\frac{\forall x. P(x)}{P(c)} \quad (\text{for any } c)$$

- If we know "All CS classes at W&M are hard", universal instantiation allows us to conclude "CS243 is hard"!

Example

- Consider predicates man(x) and mortal(x) and the hypotheses:
  1. All men are mortal:
  2. Socrates is a man:
- Using rules of inference, prove mortal(Socrates)

Universal Generalization

- Suppose we can prove a claim for an arbitrary element in the domain.
- Since we've made no assumptions about this element, proof should apply to all elements in the domain.
- This correct reasoning is captured by universal generalization

$$\frac{P(c) \text{ for arbitrary c}}{\forall x. P(x)}$$

Example

Prove $\forall x. Q(x)$ from the hypotheses:

1. $\forall x.\ (P(x) \rightarrow Q(x))$ Hypothesis
2. $\forall x.\ P(x)$ Hypothesis

Caveat About Universal Generalization

- When using universal generalization, need to ensure that c is truly arbitrary!
- If you prove something about a specific person Mary, you cannot make generalizations about all people
- In a proof, this means c must be a fresh name not used previously
- $$\frac{\text{even}(2)}{\forall x. \text{even}(x)}$$

Existential Instantiation

- Consider formula $\exists x. P(x)$.
- We know there is some element, say c, in the domain for which $P(c)$ is true.
- This is called existential instantiation:

$$\frac{\exists x. P(x)}{P(c)} \text{ (for unused c)}$$

- Here, c is a fresh name (i.e., not used before in proof).
  - Otherwise, can prove non-sensical things such as: "There exists some animal that can fly. Thus, rabbits can fly"!

Example Using Existential Instantiation

Consider the hypotheses $\exists x. P(x)$ and $\forall x. \neg P(x)$. Prove that we can derive a contradiction (i.e., false) from these hypotheses.

1. $\exists x. P(x)$ Hypothesis
2. $\forall x. \neg P(x)$ Hypothesis
3.
4.
5.
6.

Existential Generalization

- Suppose we know $P(c)$ is true for some constant c
- Then, there exists an element for which P is true
- Thus, we can conlude $\exists x.P(x)$
- This inference rule called existential generalization:

$$\frac{P(c)}{\exists x.P(x)}$$

Example Using Existential Generalization

Consider the hypotheses $atWM(George)$ and $smart(George)$.
Prove $\exists x.\ (atWM(x) \land smart(x))$

1. $atWM(George)$ Hypothesis
2. $smart(George)$ Hypothesis
3.
4.

Summary of Inference Rules for Quantifiers

Name	Rule of Inference
Universal Instantiation	$\frac{\forall x.P(x)}{P(c)}$ (any c)
Universal Generalization	$\frac{P(c) \quad \text{(for arbitrary } c)}{\forall x.P(x)}$
Existential Instantiation	$\frac{\exists x.P(x)}{P(c) \text{ for fresh } c}$
Existential Generalization	$\frac{P(c)}{\exists x.P(x)}$

Example I

- Prove that these hypotheses imply $\exists x.(P(x) \land \neg B(x))$:
  1. $\exists x.\ (C(x) \land \neg B(x))$ (Hypothesis)
  2. $\forall x.\ (C(x) \rightarrow P(x))$ (Hypothesis)

Example II

- Prove the below hypotheses are contradictory by deriving false
  1. $\forall x.(P(x) \rightarrow (Q(x) \land S(x)))$ (Hypothesis)
  2. $\forall x.(P(x) \land R(x))$ (Hypothesis)
  3. $\exists x.(\neg R(x) \lor \neg S(x))$ (Hypothesis)