CS311H: Discrete Mathematics

Divide-and-Conquer Algorithms and The Master Theorem

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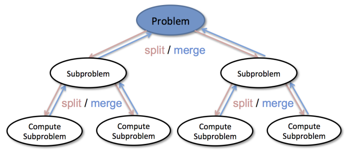
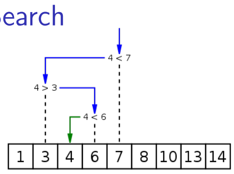
Divide-and-Conquer Algorithms



- Divide-and-conquer algorithms are recursive algorithms that:
  1. Divide problem into k smaller subproblems of the same form
  2. Solve the subproblems
  3. Conquer the original problem by combining solutions of subproblems

Example I: Binary Search



- Problem: Given sorted array of integers, is i in the array?
- Binary search algorithm:
  1. Compare i with middle element m of array
  2. If $i > m$, then recursively search right half
  3. Otherwise, recursively search left half
- Classic divide-and-conquer algorithm

Binary Search, cont.

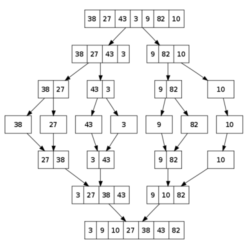
- Question: What is the worst-case complexity of binary search?
- Let $T(n)$ denote # of steps taken on input array of size n
- Write recurrence relation for $T(n)$:
- Initial condition:
- How do we get a Big-O estimate from this recurrence?
- Idea: Solve the recurrence and then find Big-O estimate for it

Solving Recurrence for Binary Search

$$T(n) = T(\frac{n}{2}) + 1 \qquad T(1) = 1$$

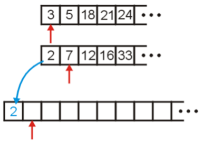
- Not in a form we can immediately solve, but can massage it!
- Let $n = 2^k$: $T(2^k) = T(2^{k-1}) + 1$
- Now, let $a_k = T(2^k)$: $a_k = a_{k-1} + 1 \quad a_0 = 1$
- What's the solution for this recurrence?
- Since $n = 2^k$, this implies $T(n) = \log_2 n + 1$
- Hence, complexity of binary search: $\Theta(\log\ n)$

Example II: Merge Sort



- Problem: Sort elements in array
- Merge sort solution:
  1. Recursively sort left half of array
  2. Recursively sort right half of array
  3. Merge the two sorted arrays

How to Merge Two Sorted Arrays?



- Input: Two sorted arrays A_1, A_2
- Output: New sorted array that includes all elements in A_1, A_2
- Idea: Pointers to current elements in A_1, A_2 (initially first)
- Copy smaller element to output array and advance pointer
- If combined size of A_1, A_2 is n, merging takes $4n$ steps (compare, advance two pointers, copy)

Recurrence Relation for Merge Sort

- What is worst-case complexity of Merge Sort?
- Let $T(n)$ be # operations performed to sort array of length n
- What is a recurrence relation for $T(n)$?
- As before, let $n = 2^k$:
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Solving Recurrence Relation

$$a_k = 2 \cdot a_{k-1} + 4 \cdot 2^k \quad a_0 = 1$$

- Particular solution form:
- Particular solution:
- Solution for homogeneous recurrence:
- Solve for α: $\alpha \cdot 2^0 + 0 \cdot 2^2 = 1 \Rightarrow \alpha = 1$
- Solution:
- Plug in $k = \log_2 n$:
- Hence, algorithm is $\Theta(n \cdot \log n)$

Summary

- Recurrence relations for divide-conquer algorithms look like:

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

- These are called divide-and-conquer recurrence relations
- To determine complexity of a divide-and conquer algorithm:
  1. Write corresponding recurrence relation
  2. Solve it exactly
  3. Obtain Θ estimate
- **Can we obtain a Θ estimate without solving recurrence exactly?**

The Master Theorem

Consider the recurrence $T(n) = a \cdot T(\frac{n}{b}) + c \cdot n^d$ where $a, c \geq 1$, $d \geq 0$, and $b > 1$. Then:

1. $T(n)$ is $\Theta(n^d)$ if $a < b^d$
2. $T(n)$ is $\Theta(n^d \log n)$ if $a = b^d$
3. $T(n)$ is $\Theta(n^{\log_b a})$ if $a > b^d$

Revisiting Examples

- Example 1: Recurrence for binary search: $T(n) = T(\frac{n}{2}) + 1$
- Here, $a = 1, b = 2, d = 0$, Hence $a = b^d$
- By Case 2 of Master Thm, $T(n) = \Theta(n^0 log\ n) = \Theta(\log n)$
- Example 2: Recurrence for merge sort: $T(n) = 2 \cdot T(\frac{n}{2}) + 4n$
- Here, $a = 2, b = 2, d = 1$, Hence $a = b^d$
- By Case 2 of Master Thm, $T(n) = \Theta(n \cdot \log n)$

More Examples

- Example 3: Consider recurrence $T(n) = 2 \cdot T(\frac{n}{2}) + 3$
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- Example 4: Consider recurrence $T(n) = T(\frac{n}{2}) + n^2$
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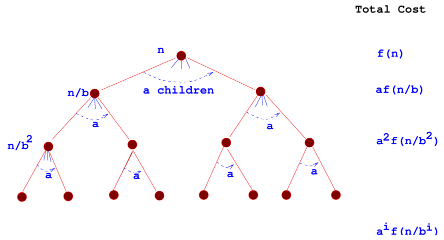
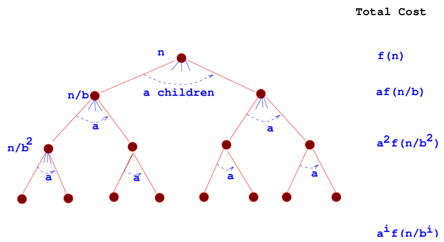
Why is the Master Theorem True?

Consider the recurrence $T(n) = a \cdot T(\frac{n}{b}) + c \cdot n^d$

- At every level of recursion, # subproblems multiplied by a
- But size of subproblem divided by b
- Let $f(n)$ be $c \cdot n^d$

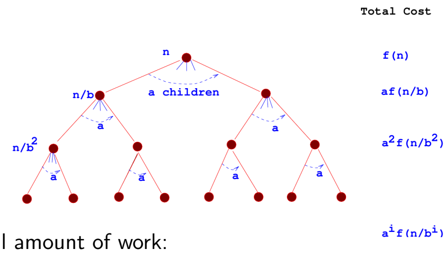


Proof of Master Theorem



- What is the height h of this tree?
- Since problem size is 1 in base case, $\frac{n}{b^h} = 1 \Rightarrow h = \log_b n$
- At the i'th level, we have a^i subproblems, hence $a^{\log_b n}$ leaves
- Equal to $n^{\log_b a}$ – verify by taking $\log_b$ of both sides

Proof of Master Theorem, cont.



- Total amount of work:

$$T(n) = \Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} a^i \cdot c \cdot (\frac{n}{b^i})^d$$

- Can be rewritten as:

$$T(n) = \Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} c \cdot (\frac{a}{b^d})^i \cdot n^d$$

Proof of Master Theorem, cont.

$$T(n) = \Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} c \cdot (\frac{a}{b^d})^i \cdot n^d$$

- Case 1: $\frac{a}{b^d} < 1$. In this case, $T(n)$ is of the form:

$$T(n) = \Theta(n^{\log_b a}) + c \cdot n^d \cdot \sum_{i=0}^{\log_b n - 1} r^i \quad \text{for } |r| < 1$$

- Hence: $T(n) = \Theta(n^{\log_b a}) + \Theta(n^d)$
- Since $\frac{a}{b^d} < 1$, we have $\log_b a - d < 1$. Thus $T(n) = \Theta(n^d)$

Proof of Master Theorem, cont.

$$T(n) = \Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} c \cdot (\frac{a}{b^d})^i \cdot n^d$$

- Case 2: $a = b^d$. In this case, $T(n)$ is of the form:

$$T(n) = \Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} c \cdot n^d$$

- Hence: $T(n) = \Theta(n^{\log_b a}) + \Theta(n^d \cdot \log_b n)$
- Since $n^{\log_b a} = n^d$, this is $\Theta(n^d \cdot \log_b n)$

Proof of Master Theorem, cont.

$$T(n) = \Theta(n^{\log_b a}) + \sum_{i=0}^{\log_b n - 1} c \cdot (\frac{a}{b^d})^i \cdot n^d$$

- Case 3: $a > b^d$. In this case, $n^{\log_b a} > n^d$.
- Use closed formula for geometric series to expand summation:

$$c \cdot n^d \cdot \frac{a}{b^d} \cdot \frac{1 - (\frac{a}{b^d})^{\log_b n - 1}}{1 - \frac{a}{b^d}}$$

- This can be rewritten to $c_1 a^{\log_b n} + c_2 n^d$ for some constants c_1, c_2
- Since , $a^{\log_b n} = n^{\log_b a}$, $T(n)$ is $\Theta(n^{\log_b a})$