CS311H: Discrete Mathematics

Propositional Logic II

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Validity, Unsatisfiability

- The truth value of a propositional formula depends on truth assignments to variables
- Example: $\neg p$ evaluates to true under under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for every assignment, e.g., $p \lor \neg p$
- Such formulas are called tautologies or valid formulas
- Some formulas evaluate to false for every assignment, e.g., $p \land \neg p$
- Such formulas are called unsatisfiable formulas or contradictions

Interpretations

- To make satisfiability/validity precise, we'll define interpretation of formula
- An interpretation I for a formula F is a mapping from each propositional variables in F to exactly one truth value

$$I : \{p \mapsto \text{true}, q \mapsto \text{false}, \cdots\}$$

- Each interpretation corresponds to one row in the truth table, so 2^n possible interpretations

Entailment

- Under an interpretation, every propositional formula evaluates to T or F

Formula F + Interpretation I = Truth value

- We write $I \models F$ if F evaluates to true under I
- Similarly, $I \not\models F$ if F evaluates to false under I.
- Theorem: $I \models F$ if and only if $I \not\models \neg F$

Examples

- Consider the formula $F: \ p \land q \rightarrow \neg p \lor \neg q$
- Let I_1 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{false}]$
- What does F evaluate to under I_1?
- Thus, $I_1 \models F$
- Let I_2 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$
- What does F evaluate to under I_2?
- Thus, $I_2 \not\models F$

Another Example

- Let F_1 and F_2 be two propositional formulas
- Suppose F_1 evaluates to true under interpretation I
- What does $F_2 \land \neg F_1$ evaluate to under I?

Satisfiability, Validity

- F is satisfiable iff there exists interpretation I s.t. $I \models F$
- F is valid iff for all interpretations I, $I \models F$
- F is unsatisfiable iff for all interpretations I, $I \not\models F$
- F is contingent if it is satisfiable, but not valid.

True/False Questions

Are the following statements true or false?

- If a formula is valid, then it is also satisfiable.
- If a formula is satisfiable, then its negation is unsatisfiable.
- If F_1 and F_2 are satisfiable, then $F_1 \wedge F_2$ is also satisfiable.
- If F_1 and F_2 are satisfiable, then $F_1 \vee F_2$ is also satisfiable.

Duality Between Validity and Unsatisfiability

F is valid if and only if $\neg F$ is unsatisfiable

- Proof:

Proving Validity

- Question: How can we prove that a propositional formula is a tautology?
- Exercise: Which formulas are tautologies? Prove your answer.
  1. $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
  2. $(p \wedge q) \vee \neg p$

Proving Satisfiability, Unsatisfiability, Contingency

- Similarly, can prove satisfiability, unsatisfiability, contingency using truth tables:
  - Satisfiable: There exists a row where formula evaluates to true
  - Unsatisfiable: In all rows, formula evaluates to false
  - Contingent: Exists a row where formula evaluates to true, and another row where it evaluates to false

Exercise

- Is $(p \rightarrow q) \rightarrow (q \rightarrow p)$ valid, unsatisfiable, or contingent? Prove your answer.

Implication

- Formula F_1 implies F_2 (written $F_1 \Rightarrow F_2$) iff for all interpretations I, $I \models F_1 \rightarrow F_2$

$$F_1 \Rightarrow F_2 \text{ iff } F_1 \rightarrow F_2 \text{ is valid}$$

- Caveat: $F_1 \Rightarrow F_2$ is not a propositional logic formula; $\Rightarrow$ is not part of PL syntax!
- Instead, $F_1 \Rightarrow F_2$ is a semantic judgment, like satisfiability!

Example

- Does $p \lor q$ imply p? Prove your answer.

Equivalence

- Two formulas F_1 and F_2 are equivalent if they have same truth value for every interpretation, e.g., $p \lor p$ and p
- More precisely, formulas F_1 and F_2 are equivalent, written $F_1 \equiv F_2$ or $F_1 \Leftrightarrow F_2$, iff:

$$F_1 \Leftrightarrow F_2 \text{ iff } F_1 \leftrightarrow F_2 \text{ is valid}$$

- $\equiv, \Leftrightarrow$ not part of PL syntax; they are semantic judgments!

Example

- Prove that $p \rightarrow q$ and $\neg p \lor q$ are equivalent

Important Equivalences

- Some important equivalences are useful to know!
- Law of double negation: $\neg\neg p \equiv p$
- Identity Laws: $p \land T \equiv p \qquad p \lor F \equiv p$
- Domination Laws: $p \lor T \equiv T \qquad p \land F \equiv F$
- Idempotent Laws: $p \lor p \equiv p \qquad p \land p \equiv p$
- Negation Laws: $p \land \neg p \equiv F \quad p \lor \neg p \equiv T$

Commutativity and Distributivity Laws

- Commutative Laws: $p \lor q \equiv q \lor p \qquad p \land q \equiv q \land p$
- Distributivity Law #1: $(p \lor (q \land r)) \equiv (p \lor q) \land (p \lor r)$
- Distributivity Law #2: $(p \land (q \lor r)) \equiv (p \land q) \lor (p \land r)$
- Associativity Laws: $p \lor (q \lor r) \equiv (p \lor q) \lor r$
  $p \land (q \land r) \equiv (p \land q) \land r$
- Absorption #1: $p \land (p \lor q) \equiv p$
- Absorption #2: $p \lor (p \land q) \equiv p$

De Morgan's Laws

- Let cs311 be the proposition "John took CS311" and cs312 be the proposition "John took CS312"
- In simple English what does $\neg(cs311 \land cs312)$ mean?
- DeMorgan's law expresses exactly this equivalence!
- De Morgan's Law #1: $\neg(p \land q) \equiv (\neg p \lor \neg q)$
- De Morgan's Law #2: $\neg(p \lor q) \equiv (\neg p \land \neg q)$

Why are These Equivalences Useful?

- Use known equivalences to prove that two formulas are equivalent by rewriting one formula to another
- Examples: Prove following formulas are equivalent:
  1. $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$
  2. $\neg(p \to q)$ and $p \land \neg q$

Formalizing English Arguments in Logic

- We can use logic to prove correctness of English arguments.
- For example, consider the argument:
  - If Joe drives fast, he gets a speeding ticket.
  - Joe did not get a ticket.
  - Therefore, Joe did not drive fast.
- Let f be the proposition "Joe drives fast", and t be the proposition "Joe gets a ticket"
- How do we encode this argument as a logical formula?

Example, cont

"If Joe drives fast, he gets a speeding ticket. Joe did not get a ticket. Therefore, he did not drive fast.": $((f \to t) \land \neg t) \to \neg f$

- How can we prove this argument is valid?
- Can do this in two ways:
  1. Use truth table to show formula is tautology
  2. Use known equivalences to rewrite formula to true
- Let's use equivalences

Another Example

- Can also use to logic to prove an argument is not valid.
- Suppose your friend George make the following argument:
  - If Jill carries an umbrella, it is raining.
  - Jill is not carrying an umbrella.
  - Therefore it is not raining.
- Let's use logic to prove George's argument doesn't hold water.
- Let u = "Jill is carrying an umbrella", and r = "It is raining"
- How do we encode this argument in logic?

Example, cont.

"If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore it is not raining.": $((u \to r) \land \neg u) \to \neg r$

- How can we prove George's argument is invalid?

Summary

- A formula is valid if it is true for all interpretations.
- A formula is satisfiable if it is true for at least one interpretation.
- A formula is unsatisfiable if it is false for all interpretations.
- A formula is contingent if it is true in at least one interpretation, and false in at least one interpretation.
- Two formulas F_1 and F_2 are equivalent, written $F_1 \equiv F_2$, if $F_1 \leftrightarrow F_2$ is valid