CS311H: Discrete Mathematics

Introduction to First-Order Logic

Instructor: Işıl Dillig

Why First-Order Logic?

- So far, we studied the simplest logic: propositional logic
- But for some applications, propositional logic is not expressive enough
- First-order logic is more expressive: allows representing more complex facts and making more sophisticated inferences

A Motivating Example

- For instance, consider the statement "Anyone who drives fast gets a speeding ticket"
- From this, we should be able to conclude "If Joe drives fast, he will get a speeding ticket"
- Similarly, we should be able to conclude "If Rachel drives fast, she will get a speeding ticket" and so on.
- But PL does not allow inferences like that because we cannot talk about concepts like "everyone", "someone" etc.
- First-order logic (predicate logic) allows making such kinds of inferences

Building Blocks of First-Order Logic

- The building blocks of propositional logic were propositions
- In first-order logic, there are three kinds of basic building blocks: constants, variables, predicates
- Constants: refer to specific objects (in a universe of discourse)
- Examples: George, 6, Austin, CS311, ...
- Variables: range over objects (in a universe of discourse)
- Examples: x,y,z, ...
- If universe of discourse is cities in Texas, x can represent Houston, Austin, Dallas, San Antonio, ...

Building Blocks of First-Order Logic, cont.

- Predicates describe properties of objects or relationships between objects
- Examples: ishappy, betterthan, loves, $>$ ...
- Predicates can be applied to both constants and variables
- Examples: ishappy(George), betterthan(x,y), loves(George, Rachel), $x > 3$, ...
- A predicate $P(c)$ is true or false depending on whether property P holds for c
- Example: ishappy(George) is true if George is happy, but false otherwise

Predicate Examples

- Consider predicate even which represents if a number is even
- What is truth value of even(2)?
- What is truth value of even(5)?
- What is truth value of even(x)?
- Another example: Suppose $Q(x, y)$ denotes $x = y + 3$
- What is the truth value of $Q(3, 0)$?
- What is the truth value of $Q(1, 2)$?

Formulas in First Order Logic

- Formulas in first-order logic are formed using predicates and logical connectives.
- Example: even(2) is a formula
- Example: even(x) is also a formula
- Example: even(x) $\vee$ odd(x) is also a formula
- Example: (odd(x) $\rightarrow \neg$ even(x)) $\wedge$ even(x)

Semantics of First-Order Logic

- In propositonal logic, the truth value of formula depends on a truth assignment to variables.
- In FOL, truth value of a formula depends interpretation of predicate symbols and variables over some domain D
- Consider a FOL formula $\neg P(x)$
- A possible interpretation:

$$D = \{\star, \circ\}, P(\star) = \text{true}, P(\circ) = \text{false}, x = \star$$

- Under this interpretation, what's truth value of $\neg P(x)$?
- What about if $x = \circ$?

More Examples

- Consider interpretation I over domain $D = \{1, 2\}$
  - $P(1,1) = P(1,2) = \text{true}, \ P(2,1) = P(2,2) = \text{false}$
  - $Q(1) = \text{false}, \ Q(2) = \text{true}$
  - $x = 1, \ y = 2$
- What is truth value of $P(x, y) \wedge Q(y)$ under I?
- What is truth value of $P(y, x) \rightarrow Q(y)$ under I?
- What is truth value of $P(x, y) \rightarrow Q(x)$ under I?

Quantifiers

- Real power of first-order logic over propositional logic: quantifiers
- Quantifiers allow us to talk about all objects or the existence of some object
- There are two quantifiers in first-order logic:
  1. Universal quantifier ($\forall$): refers to all objects
  2. Existential quantifier ($\exists$): refers to some object

Universal Quantifiers

- Universal quantification of $P(x)$, $\forall x.P(x)$, is the statement "P(x) holds for all objects x in the universe of discourse."
- $\forall x.P(x)$ is true if predicate P is true for every object in the universe of discourse, and false otherwise
- Consider domain $D = \{\circ, \star\}$, $P(\circ) = \text{true}, P(\star) = \text{false}$
- What is truth value of $\forall x.P(x)$?
- Object o for which $P(o)$ is false is counterexample of $\forall x.P(x)$
- What is a counterexample for $\forall x.P(x)$ in previous example?

More Universal Quantifier Examples

- Consider the domain D of real numbers and predicate $P(x)$ with interpretation $x^2 \geq x$
- What is the truth value of $\forall x.P(x)$?
- What is a counterexample?
- What if the domain is integers?
- Observe: Truth value of a formula depends on a universe of discourse!

Existential Quantifiers

- Existential quantification of $P(x)$, written $\exists x.P(x)$, is "There exists an element x in the domain such that $P(x)$".
- $\exists x.P(x)$ is true if there is at least one element in the domain such that $P(x)$ is true
- In first-order logic, domain is required to be non-empty.
- Consider domain $D = \{\circ, \star\}$, $P(\circ) = \text{true}, P(\star) = \text{false}$
- What is truth value of $\exists x.P(x)$?

Existential Quantifier Examples

- Consider the domain of reals and predicate $P(x)$ with interpretation $x < 0$.
- What is the truth value of $\exists x.P(x)$?
- What if domain is positive integers?
- Let $Q(y)$ be the statement $y > y^2$
- What's truth value of $\exists y.Q(y)$ if domain is reals?
- What about if domain is integers?

Quantifiers Summary

Statement	When True?	When False?
$\forall x.P(x)$	$P(x)$ is true for every x	$P(x)$ is false for some x
$\exists x.P(x)$	$P(x)$ is true for some x	$P(x)$ is false for every x

- Consider finite universe of discourse with objects $o_1, \ldots, o_n$
- $\forall x.P(x)$ is true iff $P(o_1) \land P(o_2) \ldots \land P(o_n)$ is true
- $\exists x.P(x)$ is true iff $P(o_1) \lor P(o_2) \ldots \lor P(o_n)$ is true

Quantified Formulas

- So far, only discussed how to quantify individual predicates.
- But we can also quantify entire formulas containing multiple predicates and logical connectives.
- $\exists x.(\text{even}(x) \land \text{gt}(x, 100))$ is a valid formula in FOL
- What's truth value of this formula if domain is all integers?
  - assuming $\text{even}(x)$ means "x is even" and $\text{gt}(x, y)$ means $x > y$
- What about $\forall x.(\text{even}(x) \land \text{gt}(x, 100))$?

More Examples of Quantified Formulas

- Consider the domain of integers and the predicates even(x) and div4(x) which represents if x is divisible by 4
- What is the truth value of the following quantified formulas?
  - $\forall x.\ (div4(x) \rightarrow even(x))$
  - $\forall x.\ (even(x) \rightarrow div4(x))$
  - $\exists x.\ (\neg div4(x) \land even(x))$
  - $\exists x.\ (\neg div4(x) \rightarrow even(x))$
  - $\forall x.\ (\neg div4(x) \rightarrow even(x))$

Translating English Into Quantified Formulas

Assuming $\text{freshman}(x)$ means "x is a freshman" and $\text{inCS311}(x)$ "x is taking CS311", express the following in FOL

- Someone in CS311 is a freshman
- No one in CS311 is a freshman
- Everyone taking CS311 are freshmen
- Every freshman is taking CS311

DeMorgan's Laws for Quantifiers

- Learned about DeMorgan's laws for propositional logic:

$$\neg(p \land q) \equiv \neg p \lor \neg q$$
$$\neg(p \lor q) \equiv \neg p \land \neg q$$

- DeMorgan's laws extend to first-order logic, e.g., $\neg(even(x) \lor div4(x)) \equiv (\neg even(x) \land \neg div4(x))$
- Two new DeMorgan's laws for quantifiers:

$$\neg\forall x.P(x) \equiv \exists x.\neg P(x)$$
$$\neg\exists x.P(x) \equiv \forall x.\neg P(x)$$

- When you push negation in, $\forall$ flips to $\exists$ and vice versa

Using DeMorgan's Laws

- Expressed "Noone in CS311 is a freshman" as $\neg\exists x.(\text{inCS311}(x) \land \text{freshman}(x))$
- Let's apply DeMorgan's law to this formula:
- Using the fact that $p \rightarrow q$ is equivalent to $\neg p \lor q$, we can write this formula as:
- Therefore, these two formulas are equivalent!

Nested Quantifiers

- Sometimes may be necessary to use multiple quantifiers
- For example, can't express "Everybody loves someone" using a single quantifier
- Suppose predicate $\text{loves}(x, y)$ means "Person x loves person y"
- What does $\forall x.\exists y.\text{loves}(x, y)$ mean?
- What does $\exists y.\forall x.\text{loves}(x, y)$ mean?
- Observe: Order of quantifiers is very important!

More Nested Quantifier Examples

Using the loves(x,y) predicate, how can we say the following?

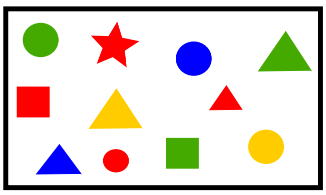
- "Someone loves everyone"
- "There is somone who doesn't love anyone"
- "There is someone who is not loved by anyone"
- "Everyone loves everyone"
- "There is someone who doesn't love herself/himself."

Summary of Nested Quantifiers

Statement	When True?
$\forall x.\forall y.P(x, y)$ $\forall y.\forall x.P(x, y)$	$P(x, y)$ is true for every pair x, y
$\forall x.\exists y.P(x, y)$	For every x, there is a y for which $P(x, y)$ is true
$\exists x.\forall y.P(x, y)$	There is an x for which $P(x, y)$ is true for every y
$\exists x.\exists y.P(x, y)$ $\exists y.\exists x.P(x, y)$	There is a pair x, y for which $P(x, y)$ is true

Observe: Order of quantifiers is only important if quantifiers of different kinds!

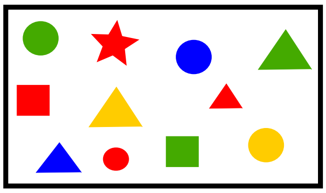
Understanding Quantifiers



Which formulas are true/false? If false, give a counterexample

- $\forall x.\exists y.\ (\text{sameShape}(x, y) \land \text{differentColor}(x, y))$
- $\forall x.\exists y.\ (\text{sameColor}(x, y) \land \text{differentShape}(x, y))$
- $\forall x.\ (\text{triangle}(x) \rightarrow (\exists y.\ (\text{circle}(y) \land \text{sameColor}(x, y))))$

Understanding Quantifiers, cont.



Which formulas are true/false? If false, give a counterexample

- $\forall x.\forall y.\ ((\text{triangle}(x) \wedge \text{square}(y)) \rightarrow \text{sameColor}(x, y))$
- $\exists x.\forall y.\neg\text{sameShape}(x, y)$
- $\forall x.\ (\text{circle}(x) \rightarrow (\exists y.(\neg\text{circle}(y) \wedge \text{sameColor}(x, y))))$

Translating First-Order Logic into English

Given predicates $student(x)$, $atUT(x)$, and $friends(x, y)$, what do the following formulas say in English?

- $\forall x.\ ((\text{atUT}(x) \wedge \text{student}(x)) \rightarrow (\exists y.(\text{friends}(x, y) \wedge \neg atUT(y))))$
- $\forall x.((\text{student}(x) \wedge \neg\text{atUT}(x)) \rightarrow \neg\exists y.\text{friends}(x, y))$
- $\forall x.\forall y.((\text{student}(x) \wedge \text{student}(y) \wedge \text{friends}(x, y)) \rightarrow (\text{atUT}(x) \wedge \text{atUT}(y)))$

Translating English into First-Order Logic

Given predicates $student(x)$, $atUT(x)$, and $friends(x, y)$, how do we express the following in first-order logic?

- "Every UT student has a friend"
- "At least one UT student has no friends"
- "All UT students are friends with each other"

Satisfiability, Validity in FOL

- The concepts of satisfiability, validity also important in FOL
- An FOL formula F is satisfiable if there exists some domain and some interpretation such that F evaluates to true
- Example: Prove that $\forall x.P(x) \rightarrow Q(x)$ is satisfiable.
- An FOL formula F is valid if, for all domains and all interpretations, F evaluates to true
- Prove that $\forall x.P(x) \rightarrow Q(x)$ is not valid.
- Formulas that are satisfiable, but not valid are contingent, e.g., $\forall x.P(x) \rightarrow Q(x)$

Equivalence

- Two formulas F_1 and F_2 are equivalent if $F_1 \leftrightarrow F_2$ is valid
- In PL, we could prove equivalence using truth tables, but not possible in FOL
- However, we can still use known equivalences to rewrite one formula as the other
- Example: Prove that $\neg(\forall x.\ (P(x) \rightarrow Q(x)))$ and $\exists x.\ (P(x) \wedge \neg Q(x))$ are equivalent.
- Example: Prove that $\neg\exists x.\forall y.P(x, y)$ and $\forall x.\exists y.\neg P(x, y)$ are equivalent.