CS311H: Discrete Mathematics

Combinatorics

Instructor: Işıl Dillig

Combinatorics

- Combinatorics (counting) deals with the question: "How many elements in a given set have desired property?"
- Counting problems can be hard $\Rightarrow$ useful to decompose
- Two basic very useful decomposition rules:
  1. Product rule: useful when task decomposes into a sequence of independent tasks
  2. Sum rule: decomposes task into a set of alternatives

Product Rule

- Suppose a task A can be decomposed into a sequence of two independent tasks B and C
- n_1 ways of doing B
- n_2 ways of doing C
- Product rule: Then, there are $n_1 n_2$ ways of doing A

Example 1

- New company with 12 offices and 2 employees Kate and Jack
- How many ways to assign different offices to Kate and Jack?
- Decomposition: First assign office to Kate, then to Jack
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Example 2

- Chairs in auditorium labeled with a letter (A-Z) and an integer $\in [1, 100]$.
- What is the max number of chairs that can be labeled?
- Observe: Max # of labeled chairs = # of different labelings
- Decomposition: First assign letter, then integer to chair
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Example 3: Extended Product Rule

- Product rule generalizes to any k tasks
- If there are n_1 ways of doing A_1, ..., n_k ways of doing A_k, then there are $n_1 \cdot n_2 \ldots \cdot n_k$ ways of doing A
- A bitstring is a string where each character is either 0 or 1
- How many different bit strings of length 7 are there?
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Counting One-to-One Functions

- How many one-to-one functions are there from a set with 3 elements to a set with 5 elements?
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Sum Rule

- Counting problems can be hard $\Rightarrow$ useful to decompose
- Two basic very useful decomposition rules:
  1. Product rule ✓
  2. Sum rule
- Suppose a task A can be done either in way B or in way C
- Suppose there are n_1 ways to do B, and n_2 ways to do C
- Sum rule: There are $n_1 + n_2$ ways to do A.

Example 1

- Suppose either a CS faculty or CS student must be chosen as representative for a committee
- There are 14 faculty, and 50 majors
- How many ways are there to choose the representative?
- By the sum rule, $50 + 14 = 64$ ways
- Note: Just like the product rule, the sum rule can be extended to more than two tasks

Example 2

- A student can choose a senior project from one of three lists
- First list contains 23 projects; second list has 15 projects, and third has 19 projects
- Also, no project appears on more than one list
- How many different projects can student choose?
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- What if some of the projects appeared on both lists?
- Caveat: For sum rule to apply, the possibilities must be mutually exclusive

More Complex Counting Problems

- Problems so far required either only product or only sum rule
- But more complex problems require a combination of both!
- Example: In a programming language, a variable name is a string of one or two characters.
- A character is either a letter [a-z] or a digit [0,9], and first character must be a letter.
- How many possible variable names are there?

Example, cont.

Another Example

- A password must be six to seven characters long
- A character is upper case letter or digit
- Each password must contain at least one digit
- How many possible passwords?
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Example, cont.

Example 3

- How many bitstrings are there of length 6 that do not have two consecutive 1's?
- Let $F(n)$ denote the number of bitstrings of length n that do not have two consecutive 1's
- We'll first derive a recursive equation to characterize $F(n)$
- By the sum rule, $F(n)$ is the sum of:

  1. $\#$ of n-bit strings starting with 1 not containing 11
  2. $\#$ of n-bit strings starting with 0 not containing 11

Example 3, cont.

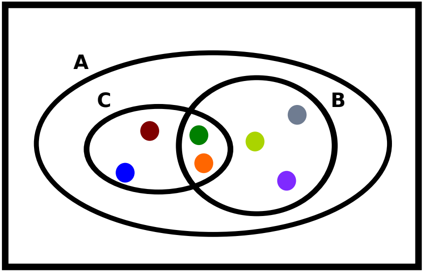
Recall: Sum Rule

- Recall: Sum rule only applies if a task is as disjunction of two mutually exclusive tasks
- What do we do if the tasks aren't mutually exclusive?
- Example: You can choose from set A or set B, but they have some elements in common
- Generalization of the sum rule: inclusion-exclusion principle

The Inclusion-Exclusion Principle

- Suppose a set A can be written as union of sets B and C
- Inclusion-Exclusion Principle:

$$|A| = |B| + |C| - |B \cap C|$$



Inclusion-Exclusion Principle Example

- How many bit strings of length 8 either start with 1 or end with two bits 00?
- Let B be the set of bitstrings that start with 1
- Let C be the set of bitstrings that end with 00
- We want $|B \cup C|$
- By the inclusion-exclusion principle, $|B \cup C| = |B| + |C| - |B \cap C|$
- Thus, compute $|B|, |C|$ and $|B \cap C|$

Example, cont.

Another Example

- A company receives 350 applications for job positions
- 220 of applicants are CS majors
- 147 of applicants are business majors
- 51 are double CS and business majors
- How many are neither CS nor business majors?

The Pigeonhole Principle

- Suppose there is a flock of 36 pigeons and a set of 35 pigeonholes
- Each pigeon want to sit in one hole
- But since there are less holes than there are pigeons, one pigeon is left without a hole.

- The Pigeonhole Principle: If $n + 1$ or more objects are placed into n boxes, then at least one box contains 2 or more objects

Examples

- Consider an event with 367 people. Is it possible no pair of people have the same birthday?
- Consider function f from a set with $k+1$ or more elements to a set with k elements. Is it possible f is one-to-one?
- Consider n married couples. How many of the $2n$ people must be selected to guarantee there is at least one married couple?

Generalized Pigeonhole Principle

- If n objects are placed into k boxes, then there is at least one box containing at least $\lceil n/k \rceil$ objects

- Proof: (by contradiction) Suppose every box contains less than $\lceil n/k \rceil$ objects

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Examples

- If there are 30 students in a class, at least how many must be born in the same month? $\left\lceil \frac{30}{12} \right\rceil = 3$
- What is the minimum # of students required to ensure at least 6 students receive the same grade (A, B, C, D, F)?
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- What is the min # of cards that must be chosen to guarantee three have same suit?
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