CS311H: Discrete Mathematics

Permutations and Combinations

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Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects
  - No object can be selected more than once
  - Order of arrangement matters
- Example: \( S = \{a, b, c\}\). What are the permutations of \( S\)?

How Many Permutations?

- Consider set \( S = \{a_1, a_2, \ldots, a_n\}\)
- How many permutations of \( S\) are there?
- Decompose using product rule:
  - How many ways to choose first element?
  - How many ways to choose second element?
  - ... 
  - How many ways to choose last element?
- What is number of permutations of set \( S\)?

Examples

- Consider the set \( \{7, 10, 23, 4\}\). How many permutations?
- How many permutations of letters A, B, C, D, E, F, G contain "ABC" as a substring?

r-Permutations

- \( r \)-permutation is ordered arrangement of \( r \) elements in a set \( S \)
  - \( S \) can contain more than \( r \) elements
  - But we want arrangement containing \( r \) of the elements in \( S \)
- The number of \( r \)-permutations in a set with \( n \) elements is written \( P(n, r) \)
- Example: What is \( P(n, n) \)?

Computing \( P(n, r) \)

- Given a set with \( n \) elements, what is \( P(n, r) \)?
- Decompose using product rule:
  - How many ways to pick first element?
  - How many ways to pick second element?
  - How many ways to pick \( i \)’th element?
  - How many ways to pick last element?
- Thus, \( P(n, r) = n \cdot (n - 1) \cdots (n - r + 1) = \frac{n!}{(n-r)!} \)
Examples

- What is the number of 2-permutations of set \{a, b, c, d, e\}?

- How many ways to select first-prize winner, second-prize winner, third-prize winner from 10 people in a contest?

- Salesman must visit 4 cities from list of 10 cities: Must begin in Chicago, but can choose the remaining cities and order. How many possible itinerary choices are there?

Combinations

- An \(r\)-combination of set \(S\) is the unordered selection of \(r\) elements from that set.
  - Unlike permutations, order does not matter in combinations.

- Example: What are 2-combinations of the set \{a, b, c\}?

- For this set, 6 2-permutations, but only 3 2-combinations.

Number of \(r\)-combinations

- The number of \(r\)-combinations of a set with \(n\) elements is written \(C(n, r)\).

- \(C(n, r)\) is often also written as \(\binom{n}{r}\), read "n choose r".

- \(\binom{n}{r}\) is also called the binomial coefficient.

- Theorem:
  \[
  C(n, r) = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}.
  \]

Proof of Theorem

- What is the relationship between \(P(n, r)\) and \(C(n, r)\)?

- Let’s decompose \(P(n, r)\) using product rule:
  - First choose \(r\) elements.
  - Then, order these \(r\) elements.

- How many ways to choose \(r\) elements from \(n\)?

- How many ways to order \(r\) elements?

- Thus, \(P(n, r) = C(n, r) \cdot r!\).

- Therefore,
  \[
  C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)! \cdot r!}.
  \]

Examples

- How many hands of 5 cards can be dealt from a standard deck of 52 cards?

- There are 9 faculty members in a math department, and 11 in CS department.

- If we must select 3 math and 4 CS faculty for a committee, how many ways are there to form this committee?

More Complicated Example

- How many bitstrings of length 8 contain at least 6 ones?
One More Example

- How many bitstrings of length 8 contain at least 3 ones and 3 zeros?

Binomial Coefficients

- Recall: $C(n, r)$ is also denoted as $\binom{n}{r}$ and is called the binomial coefficient.
- Binomial is polynomial with two terms, e.g., $(a + b), (a + b)^2$
- $\binom{n}{r}$ called binomial coefficient b/c it occurs as coefficients in the expansion of $(a + b)^n$

An Example

- Consider expansion of $(a + b)^3$
- $(a + b)^3 = (a + b)(a + b)(a + b)$
- $= (a^2 + 2ab + b^2)(a + b)$
- $= (a^3 + 3a^2b + 3ab^2 + b^3)$
- $1 \quad 3 \quad 3 \quad 1$

The Binomial Theorem

- Let $x, y$ be variables and $n$ a non-negative integer. Then,
- $(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$
- What is the expansion of $(x + y)^4$?

Another Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Corollary of Binomial Theorem

- Binomial theorem allows showing a bunch of useful results.
- Corollary: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
**Another Corollary**

- Corollary: \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \)

**Pascal’s Triangle**

- Pascal arranged binomial coefficients as a triangle
- \( n \)’th row consists of \( \binom{n}{k} \) for \( k = 0, 1, \ldots, n \)

**Proof of Pascal’s Identity**

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

- This identity is known as Pascal’s identity
- Proof:

\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)! (n-k+1)!} + \frac{n!}{k! (n-k)!} = \frac{k! n! + (n-k+1)! n!}{k! (n-k+1)!} = \frac{(n+1)!}{k! (n-k+1)!} = \binom{n+1}{k}
\]

- But this is exactly \( \binom{n+1}{k} \)

**Interesting Facts about Pascal’s Triangle**

- What is the sum of numbers in \( n \)’th row in Pascal’s triangle (starting at \( n = 0 \))?
- Observe: This is exactly the corollary we proved earlier!

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

**Some Fun Facts about Pascal’s Triangle, cont.**

- Pascal’s triangle is perfectly symmetric
- Numbers on left are mirror image of numbers on right
- Why is this the case?
Permutations with Repetitions

- Earlier, when we defined permutations, we only allowed each object to be used once in the arrangement.
- But sometimes makes sense to use an object multiple times.
- Example: How many strings of length 4 can be formed using letters in English alphabet?
- A permutation with repetition of a set of objects is an ordered arrangement of these objects, where each object may be used more than once.

General Formula for Permutations with Repetition

- $P^*(n, r)$ denotes number of $r$-permutations with repetition from set with $n$ elements.
- What is $P^*(n, r)$?
- How many ways to assign 3 jobs to 6 employees if every employee can be given more than one job?
- How many different 3-digit numbers can be formed from 1, 2, 3, 4, 5?