CS311H: Discrete Mathematics

Permutations and Combinations

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Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects
  - No object can be selected more than once
  - Order of arrangement matters
- Example: $S = \{a, b, c\}$. What are the permutations of S?
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How Many Permutations?

- Consider set $S = \{a_1, a_2, \dots a_n\}$
- How many permutations of S are there?
- Decompose using product rule:
  - How many ways to choose first element?
  - How many ways to choose second element?
  - ...
  - How many ways to choose last element?
- What is number of permutations of set S?

Examples

- Consider the set $\{7, 10, 23, 4\}$. How many permutations?
- How many permutations of letters A, B, C, D, E, F, G contain "ABC" as a substring?
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r-Permutations

- r-permutation is ordered arrangment of r elements in a set S
  - S can contain more than r elements
  - But we want arrangement containing r of the elements in S
- The number of r-permutations in a set with n elements is written $P(n, r)$
- Example: What is $P(n, n)$?

Computing $P(n, r)$

- Given a set with n elements, what is $P(n, r)$?
- Decompose using product rule:
  - How many ways to pick first element?
  - How many ways to pick second element?
  - How many ways to pick i'th element?
  - How many ways to pick last element?
- Thus, $P(n, r) = n \cdot (n-1) \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$

Examples

- What is the number of 2-permutations of set $\{a, b, c, d, e\}$?
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- How many ways to select first-prize winner, second-prize winner, third-prize winner from 10 people in a contest?
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- Salesman must visit 4 cities from list of 10 cities: Must begin in Chicago, but can choose the remaining cities and order.
- How many possible itinerary choices are there?
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Combinations

- An r-combination of set S is the unordered selection of r elements from that set
  - Unlike permutations, order does not matter in combinations
- Example: What are 2-combinations of the set $\{a, b, c\}$?
- For this set, 6 2-permutations, but only 3 2-combinations

Number of r-combinations

- The number of r-combinations of a set with n elements is written $C(n, r)$
- $C(n, r)$ is often also written as $\left(\begin{array}{c} n \\ r \end{array} \right)$, read "n choose r"
- $\left(\begin{array}{c} n \\ r \end{array} \right)$ is also called the binomial coefficient
- Theorem:

$$C(n, r) = \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r! \cdot (n-r)!}$$

Proof of Theorem

- What is the relationship between $P(n, r)$ and $C(n, r)$?
- Let's decompose $P(n, r)$ using product rule:
  - First choose r elements
  - Then, order these r elements
- How many ways to choose r elements from n?
- How many ways to order r elements?
- Thus, $P(n, r) = C(n, r) * r!$
- Therefore,

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)! \cdot r!}$$

Examples

- How many hands of 5 cards can be dealt from a standard deck of 52 cards?
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- There are 9 faculty members in a math department, and 11 in CS department.
- If we must select 3 math and 4 CS faculty for a committee, how many ways are there to form this committee?
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More Complicated Example

- How many bitstrings of length 8 contain at least 6 ones?
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One More Example

- How many bitstrings of length 8 contain at least 3 ones and 3 zeros?
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Binomial Coefficients

- Recall: $C(n, r)$ is also denoted as $\left(\begin{array}{c} n \\ r \end{array} \right)$ and is called the binomial coefficient
- Binomial is polynomial with two terms, e.g., $(a + b), (a + b)^2$
- $\left(\begin{array}{c} n \\ r \end{array} \right)$ called binomial coefficient b/c it occurs as coefficients in the expansion of $(a + b)^n$

An Example

- Consider expansion of $(a+b)^3$
- $(a+b)^3 = (a+b)(a+b)(a+b)$
- $\qquad = (a^2+2ab+b^2)(a+b)$
- $\qquad = (a^3+2a^2b+ab^2)+(a^2b+2ab^2+b^3)$
- $\qquad = 1a^3+3a^2b+3ab^2+1b^3$

$$\begin{array}{cccc} \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \end{array}$$

The Binomial Theorem

- Let x, y be variables and n a non-negative integer. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

- What is the expansion of $(x+y)^4$?

Another Example

- What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?
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Corollary of Binomial Theorem

- Binomial theorem allows showing a bunch of useful results.
- Corollary: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
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Another Corollary

- Corollary: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$
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Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

- Pascal arranged binomial coefficients as a triangle
- n'th row consists of $\binom{n}{k}$ for $k = 0, 1, \ldots n$

Proof of Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- This identity is known as Pascal's identity
- Proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(k)!(n-k)!}$$

- Multiply first fraction by $\frac{k}{k}$ and second by $\frac{n-k+1}{n-k+1}$:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{(k)!(n-k+1)!}$$

Proof of Pascal's Identity, cont.

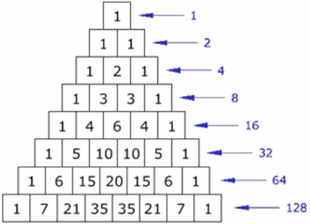
$$\binom{n}{k-1} + \binom{n}{k} = \frac{k \cdot n! + (n-k+1)n!}{(k)!(n-k+1)!}$$

- Factor the numerator:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{(n+1) \cdot n!}{(k)!(n-k+1)!} = \frac{(n+1)!}{k! \cdot (n-k+1)!}$$

- But this is exactly $\binom{n+1}{k}$

Interesting Facts about Pascal's Triangle

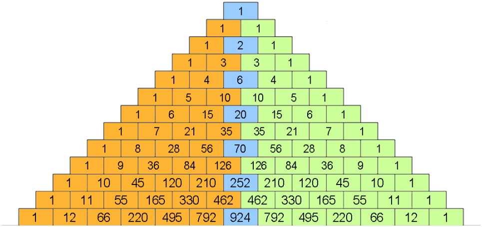


- What is the sum of numbers in n'th row in Pascal's triangle (starting at $n = 0$)?

- Observe: This is exactly the corollary we proved earlier!

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Some Fun Facts about Pascal's Triangle, cont.



- Pascal's triangle is perfectly symmetric
  - Numbers on left are mirror image of numbers on right
- Why is this the case?

Permutations with Repetitions

- Earlier, when we defined permutations, we only allowed each object to be used once in the arrangement
- But sometimes makes sense to use an object multiple times
- Example: How many strings of length 4 can be formed using letters in English alphabet?
- A permutation with repetition of a set of objects is an ordered arrangement of these objects, where each object may be used more than once

General Formula for Permutations with Repetition

- $P^*(n,r)$ denotes number of r-permutations with repetition from set with n elements
- What is $P^*(n,r)$?
- How many ways to assign 3 jobs to 6 employees if every employee can be given more than one job?
- How many different 3-digit numbers can be formed from $1, 2, 3, 4, 5$?