CS311H: Discrete Mathematics

Combinatorics 3

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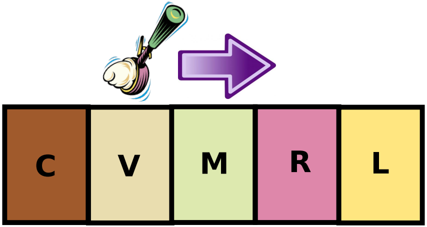
Combinations with Repetition

- Combinations help us to answer the question "In how many ways can we choose r objects from n objects?"
- Now, consider the slightly different question: "In how many ways can we choose r objects from n kinds of objects?
- These questions are quite different:
  - For first question, once we pick one of the n objects, we cannot pick the same object again
  - For second question, once we pick one of the n kinds of objects, we can pick the same type of object again!
- Combination with repetition allows answering the latter type of question!

Example

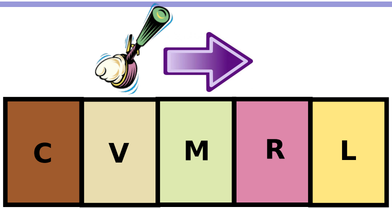
- An ice cream dessert consists of three scoops of ice cream
- Each scoop can be one of the flavors: chocolate, vanilla, mint, lemon, raspberry
- In how many different ways can you pick your dessert?
- Example of combination with repetition: "In how many ways can we pick 3 objects from 5 kinds of objects?"
- Caveat: Despite looking deceptively simple, quite difficult to figure this out (at least for me...)

Example, cont.



- To solve problem, imagine we have ice cream in boxes.
- We start with leftmost box, and proceed towards right.
- At every box, you can take 0-3 scoops, and then move to next.
- Denote taking a scoop by $\circ$ and moving to next box by $\rightarrow$

Example, cont.



- Let's look at some selections and their representation:
  - 3 scoops of chocolate: $\circ\ \circ\ \circ \rightarrow\rightarrow\rightarrow\rightarrow$
  - 1 vanilla, 1 raspberry, 1 lemon: $\rightarrow \circ \rightarrow\rightarrow \circ \rightarrow \circ$
  - 2 mint, 1 raspberry: $\rightarrow\rightarrow \circ\circ \rightarrow \circ \rightarrow$
- Invariant: r circles and $n-1$ arrows (here, $r=3, n=5$)
- Our question is equivalent to: "In how many ways can we arrange r circles and $n-1$ arrows?"

Result

- We'll denote the number of ways to choose r objects from n kinds of objects $C^*(n, r)$:

$$C^*(n, r) = \binom{n + r - 1}{r}$$

- Example: In how many ways can we choose 3 scoops of ice cream from 5 different flavors?

- Here, $r = 3$ and $n = 5$. Thus:

$$\binom{7}{3} = \frac{7!}{3! \cdot 4!} = 35$$

Example 1

- Suppose there is a bowl containing apples, oranges, and pears
  - There is at least four of each type of fruit in the bowl
- How many ways to select four pieces of fruit from this bowl?
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Example 2

- Consider a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills
  - There is at least five of each type of bill in the box
- How many ways are there to select 5 bills from this cash box?
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Example 3

- Assuming x_1, x_2, x_3 are non-negative integers, how many solutions does $x_1 + x_2 + x_3 = 11$ have?
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Example 4

- Suppose x_1, x_2, x_3 are integers s.t. $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$.
- Then, how many solutions does $x_1 + x_2 + x_3 = 11$ have?
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Summary of Different Permuations and Combinations

Order matters?	Question: How many ways to pick r objects from ...	
	n objects	n types of objects
Yes	Permutation $P(n,r) = \frac{n!}{(n-r)!}$	Permutation w/ repetition $P^*(n,r) = n^r$
No	Combination $C(n,r) = \frac{n!}{r! \cdot (n-r)!}$	Combination w/ repetition $C^*(n,r) = \frac{(n+r-1)!}{r! \cdot (n-1)!}$

Permutations with Indistinguishable Objects

- How many different strings can be made by reordering the letters in the word ALL?
- This is not given by $3!$ because some of the letters in this word are the same
- Different strings: ALL, LAL, LLA $\Rightarrow 3$ possibilities, not 6 because relative ordering of L's doesn't matter
- In general, how can we compute the number of permutations of n objects where some of them are indistinguishable?

Permutations with Indistinguishable Objects, cont.

- Consider n objects such that:
  - n_1 of them indistingishable of type 1
  - n_2 of them indistingishable of type 2
  - ...
  - n_k of them indistinguishable of type k
- The number of permutations in this case is given by:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Proof

- Let's decompose using product rule:
  - First, place all n_1 objects of type 1
  - Then all n_2 objects of type 2 etc.
- How many ways to place n_1 indistinguishable objects in n slots?
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Proof, cont.

- Now, how many ways to place n_2 objects of type 2?
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- Continuing this way and using product rule, number of permutations is:

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdots \binom{n-\sum_{i=1}^{k-1} n_i}{n_k}$$

Proof, cont.

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdots \binom{n - \sum_{i=1}^{k-1} n_i}{n_k}$$

- Let's expand this definition:

$$\frac{n!}{n_1! \cdot (n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!} \cdot \ldots \cdot \frac{(n-n_1-n_2-\ldots n_{k-1})!}{k! \cdot 0!} \quad \frac{n!}{n_1! \cdot \cancel{(n-n_1)!}} \cdot \frac{}{n_2}$$

- This simplifies to: $\dfrac{n!}{n_1! n_2! \ldots n_k!}$

- Another way to see this: Compute total # of permutations ($n!$) and then divide by # of relative orderings between objects of type 1 ($n_1!$), # of relative orderings of objects of type 2 ($n_2!$) etc.

Example 1

- How many different strings can be made by ordering the letters of the word SUCCESS?
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Example 2

- There are 3 identical red balls, 5 identical blue balls, and 2 identical green balls.
- In how many different ways can these balls be arranged if the first ball must be blue?
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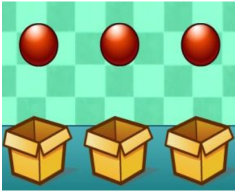
Distributing Objects into Boxes



- Many counting problems can be thought of as distributing objects into boxes
- In some cases both objects and boxes are distinguishable
- In some cases, boxes are distinguishable, but objects are indistinguishable

Example: Distinguishable Objects in Distinguishable Boxes

- How many ways are there to distribute 5 cards to each of 4 players from a deck of 52 cards?
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Example, cont.

Indistinguishable Objects Into Distinguishable Boxes

- How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?
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Distributing Objects into Boxes Summary

- Distinguishable objects into distinguishable boxes:
  - Involves permutations
- Indistinguishable objects into distinguishable boxes:
  - Involves combination

Another Example

- How many ways to distribute six distinguishable objects to five distinguishable boxes?
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Example 2

- How many ways to distribute six indistinguishable objects to five distinguishable boxes?
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Example 3

- How many ways to assign 15 distinguishable objects into 5 distinguishable boxes so boxes contain $1, 2, 3, 4, 5$ objects respectively?
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