

CS311H: Discrete Mathematics

Functions

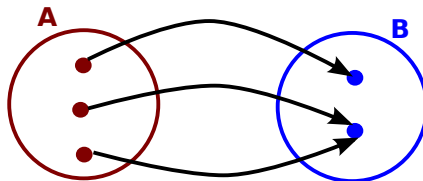
Instructor: Işıl Dillig

Functions

- ▶ A **function** f from a set A to a set B assigns each element of A to exactly one element of B .
- ▶ A is called **domain** of f , and B is called **codomain** of f .
- ▶ If f maps element $a \in A$ to element $b \in B$, we write $f(a) = b$
- ▶ If $f(a) = b$, b is called **image** of a ; a is in **preimage** of b .
- ▶ **Range** of f is the set of **all** images of elements in A .

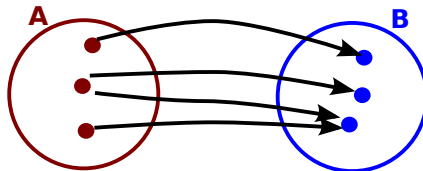
Functions Examples and Non-Examples

Is this mapping a function?



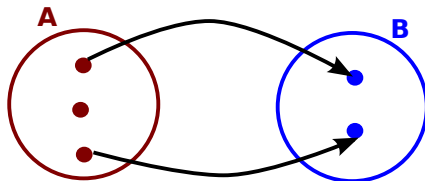
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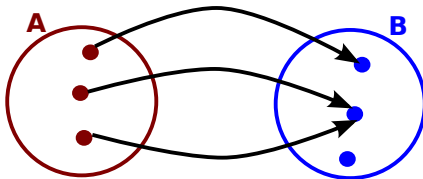
Functions Examples and Non-Examples

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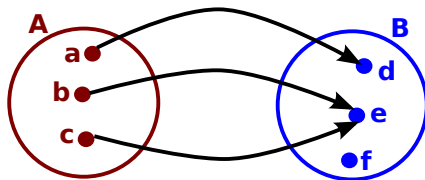


Functions Examples and Non-Examples

Is this mapping a function?



Function Terminology Examples



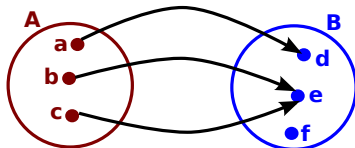
- ▶ What is the range of this function?
- ▶ What is the image of c ?
- ▶ What is the preimage of e ?

Image of a Set

- ▶ We can extend the definition of image to a set
- ▶ Suppose f is a function from A to B and S is a subset of A
- ▶ The **image** of S under f includes exactly those elements of B that are images of elements of S :

$$f(S) = \{t \mid \exists s \in S. t = f(s)\}$$

- ▶ What is the image of $\{b, c\}$?



One-to-One Functions

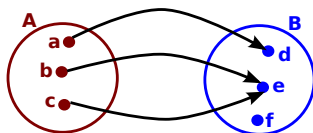
- ▶ A function f is called **one-to-one** if and only if $f(x) = f(y)$ implies $x = y$ for every x, y in the domain of f :

$$\forall x, y. (f(x) = f(y) \rightarrow x = y)$$

- ▶ One-to-one functions never assign different elements in the domain to the same element in the codomain:

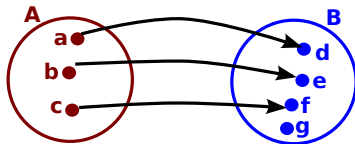
$$\forall x, y. (x \neq y \rightarrow f(x) \neq f(y))$$

- ▶ A one-to-one function also called **injection** or **injective function**
- ▶ Is this function one-to-one?



More Injective Function Examples

- Is this function injective?



- Consider the function $f(x) = x^2$ from set of integers to set of integers. Is this injective?
- What about if the domain of f is the set of non-negative integers?

Proving Injectivity Example

- ▶ Consider the function f from $\mathbb{Z}$ to $\mathbb{Z}$ defined as:

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \geq 0 \\ -3x + 2 & \text{if } x < 0 \end{cases}$$

- ▶ Prove that f is injective.
- ▶ We need to show that if $x \neq y$, then $f(x) \neq f(y)$
- ▶ What proof technique do we need to use?

Proving Injectivity Example, cont.

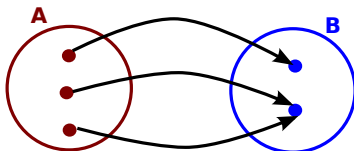
Proving Injectivity Example, cont.

Onto Functions

- ▶ A function f from A to B is called **onto** iff for every element $y \in B$, there is an element $x \in A$ such that $f(x) = y$:

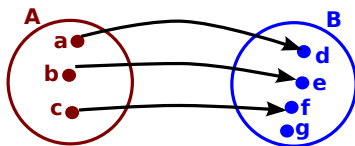
$$\forall y \in B. \exists x \in A. f(x) = y$$

- ▶ **Note:** $\exists x \in A. \phi$ is shorthand for $\exists x.(x \in A \wedge \phi)$, and $\forall x \in A. \phi$ is shorthand for $\forall x.(x \in A \rightarrow \phi)$
- ▶ Onto functions also called **surjective functions** or **surjections**
- ▶ For onto functions, range and codomain are the same
- ▶ Is this function onto?



Examples of Onto Functions

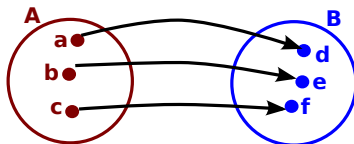
- Is this function onto?



- Consider the function $f(x) = x^2$ from the set of integers to the set of integers. Is f surjective?

Bijjective Functions

- ▶ Function that is both onto and one-to-one called **bijection**
- ▶ Bijection also called **one-to-one correspondence** or **invertible function**
- ▶ Example of bijection:



Bijection Example

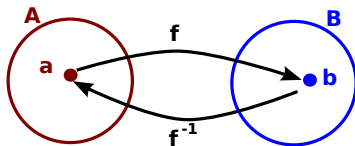
- ▶ The **identity function** I on a set A is the function that assigns every element of A to itself, i.e., $\forall x \in A. I(x) = x$
- ▶ Prove that the identity function is a bijection.
- ▶ Need to prove I is both one-to-one and onto.
- ▶ **One-to-one:** We need to show $\forall x, y. (x \neq y \rightarrow I(x) \neq I(y))$
- ▶ Suppose $x \neq y$.
- ▶ Since $I(x) = x$ and $I(y) = y$, and $x \neq y$, $I(x) \neq I(y)$

Bijection Example, cont.

- ▶ Now, prove I is **onto**, i.e., for every b , there exists some a such that $f(a) = b$
- ▶ For contradiction, suppose there is some b such that $\forall a \in A. I(a) \neq b$
- ▶ Since $I(a) = a$, this means $\forall a \in A. a \neq b$
- ▶ But since b is itself in A , this would imply $b \neq b$, yielding a contradiction.
- ▶ Since I is both onto and one-to-one, it is a bijection. □

Inverse Functions

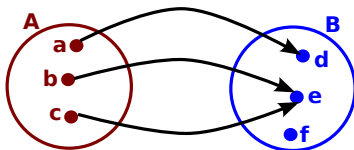
- ▶ Every bijection from set A to set B also has an **inverse function**
- ▶ The inverse of bijection f , written f^{-1} , is the function that assigns to $b \in B$ a unique element $a \in A$ such that $f(a) = b$



- ▶ **Observe:** Inverse functions are only defined for bijections, not arbitrary functions!
- ▶ This is why bijections are also called **invertible functions**

Why are Inverse Functions Only Defined on Bijections?

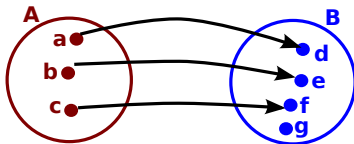
- ▶ Suppose f is not injective, i.e., assigns distinct elements to the same element.



- ▶ Then, the inverse is not a function because it assigns the same element to distinct elements

Why are Inverse Functions Only Defined on Bijections?

- ▶ Suppose f is not surjective, i.e., range and codomain are not the same



- ▶ Then, the inverse is not a function because it does not assign some element in B to any element in A
- ▶ Hence, inverse functions only defined for bijections!

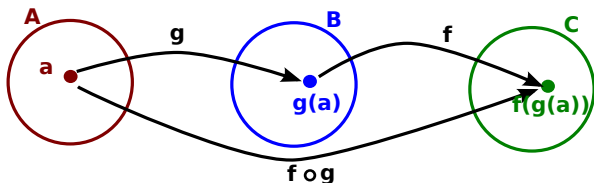
Inverse Function Examples

- ▶ Let f be the function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f(x) = x^2$. Is f invertible?
- ▶ Let g be the function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $g(x) = x + 1$. Is g invertible?

Function Composition

- ▶ Let g be a function from A to B , and f from B to C .
- ▶ The **composition** of f and g , written $f \circ g$, is defined by:

$$(f \circ g)(x) = f(g(x))$$



Composition Example

- ▶ Let f and g be function from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f(x) = 2x + 3$ and $g(x) = 3x + 2$
- ▶ What is $f \circ g$?

Another Composition Example

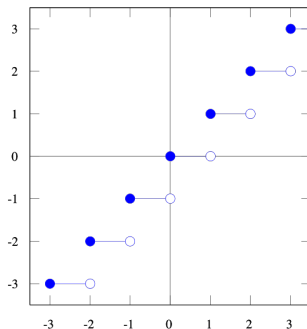
- ▶ Prove that $f^{-1} \circ f = I$ where I is the identity function.
- ▶ Since $I(x) = x$, need to show $(f^{-1} \circ f)(x) = x$
- ▶ First, $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
- ▶ Let $f(x)$ be y
- ▶ Then, $f^{-1}(f(x)) = f^{-1}(y)$
- ▶ By definition of inverse, $f^{-1}(y) = x$ iff $f(x) = y$
- ▶ Thus, $f^{-1}(f(x)) = f^{-1}(y) = x$ □

Example

- Prove that if f and g are injective, then $f \circ g$ is also injective.

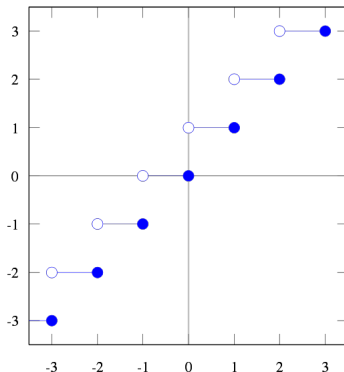
Floor and Ceiling Functions

- ▶ Two important functions in discrete math are **floor** and **ceiling** functions, both from $\mathbb{R}$ to $\mathbb{Z}$
- ▶ The **floor** of a real number x , written $\lfloor x \rfloor$, is the largest **integer** less than or equal to x .

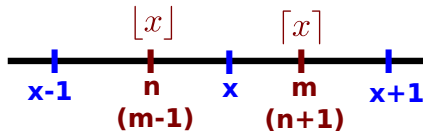


Ceiling Function

- ▶ The **ceiling** of a real number x , written $\lceil x \rceil$, is the smallest **integer** greater than or equal to x .

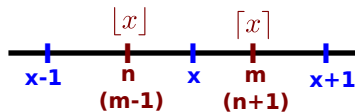


Useful Properties of Floor and Ceiling Functions



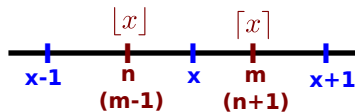
1. For integer n and real number x , $\lfloor x \rfloor = n$ iff $n \leq x < n + 1$
2. For integer n and real number x , $\lceil x \rceil = m$ iff $m - 1 < x \leq m$
3. For any real x , $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

Proofs about Floor/Ceiling Functions



Prove that $\lfloor -x \rfloor = -\lceil x \rceil$

Another Example



Prove that $\lfloor x + k \rfloor = \lfloor x \rfloor + k$ where k is an integer

More Examples

Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

- ▶ **Observe:** Any real number x can be written as $n + \epsilon$ where $n = \lfloor x \rfloor$ and $0 \leq \epsilon < 1$
- ▶ To prove desired property, do proof by cases
- ▶ **Case 1:** $0 \leq \epsilon < \frac{1}{2}$
- ▶ **Case 2:** $\frac{1}{2} \leq \epsilon < 1$
- ▶ First prove property for first case, then second case

Revisiting Sets

- ▶ Earlier we talked about sets and cardinality of sets
- ▶ Recall: **Cardinality** of a set is number of elements in that set
- ▶ This definition makes sense for sets with finitely many element, but more involved for infinite sets
- ▶ Agenda: Revisit the notion of cardinality for infinite sets

Cardinality of Infinite Sets

- ▶ Sets with infinite cardinality are classified into two classes:
 1. Countably infinite sets (e.g., natural numbers)
 2. Uncountably infinite sets (e.g., real numbers)
- ▶ A set A is called **countably infinite** if there is a **bijection** between A and the set of positive integers.
- ▶ A set A is called **countable** if it is either finite or countably infinite
- ▶ Otherwise, the set is called **uncountable** or **uncountably infinite**

Example

Prove: The set of odd positive integers is countably infinite.

- ▶ Need to find a function f from $\mathbb{Z}^+$ to the set of odd positive integers, and prove that f is bijective
- ▶ Consider $f(n) = 2n - 1$ from $\mathbb{Z}^+$ to odd positive integers
- ▶ We need to show f is bijective (i.e., one-to-one and onto)
- ▶ Let's first prove injectivity, then surjectivity

Example, cont.

Another Way to Prove Countable-ness

- ▶ One way to show a set A is countably infinite is to give bijection between $\mathbb{Z}^+$ and A
- ▶ Another way is by showing members of A can be written as a sequence $(a_1, a_2, a_3, \dots)$
- ▶ Since such a sequence is a bijective function from $\mathbb{Z}^+$ to A , writing A as a sequence $a_1, a_2, a_3, \dots$ establishes one-to-one correspondence

Another Example

Prove that the set of **all integers** is countable

- ▶ We can list all integers in a sequence, alternating positive and negative integers:

$$a_n = 0, 1, -1, 2, -2, 3, -3, \dots$$

- ▶ Observe that this sequence defines the bijective function:

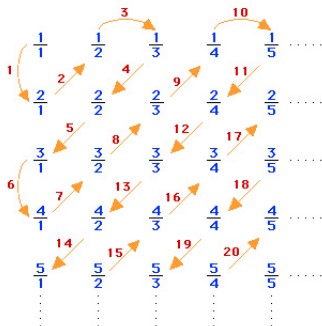
$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n-1)/2 & \text{if } n \text{ odd} \end{cases}$$

Rational Numbers are Countable

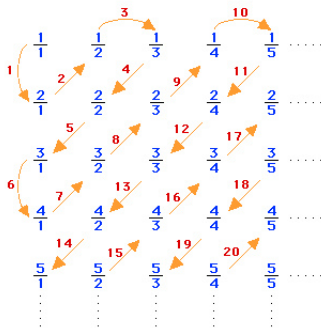
- ▶ Not too surprising $\mathbb{Z}$ and odd $\mathbb{Z}^+$ are countably infinite
- ▶ **More surprising:** Set of rationals is also countably infinite!
- ▶ We'll prove that the set of positive rational numbers is countable by showing how to enumerate them in a sequence
- ▶ **Recall:** Every positive rational number can be written as the quotient p/q of two positive integers p, q

Rationals in a Table

- Now imagine placing rationals in a table such that:
 1. Rationals with $p = 1$ go in first row, $p = 2$ in second row, etc.
 2. Rationals with $q = 1$ in 1st column, $q = 2$ in 2nd column, ...

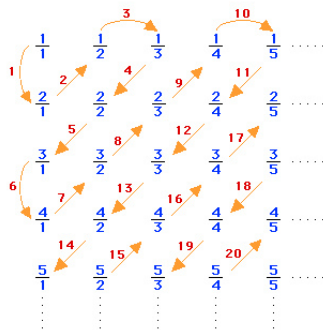


Enumerating the Rationals



- ▶ How to enumerate entries in this table without missing any?
- ▶ **Trick:** First list those with $p + q = 2$, then $p + q = 3$, ...
- ▶ Traverse table diagonally from left-to-right, in the order shown by arrows

Enumerating the Rationals, cont.



- This allows us to list all rationals in a sequence:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \dots$$

- Hence, set of rationals is countable

Uncountability of Real Numbers

- ▶ Prime example of uncountably infinite sets is **real numbers**
- ▶ The fact that $\mathbb{R}$ is uncountably infinite was proven by George Cantor using the famous **Cantor's diagonalization argument**
- ▶ Reminiscent of Russell's paradox

Cantor's Diagonalization Argument

- ▶ For contradiction, assume the set of reals was countable
- ▶ Since any subset of a countable set is also countable, this would imply the set of reals between 0 and 1 is also countable
- ▶ Now, if reals between 0 and 1 are countable, we can list them in the following way:

$$\begin{array}{rcllclcl} R_1 = & 0. & [a_{11}] & a_{12} & a_{12} & \cdots & a_{1n} & \cdots \\ R_2 = & 0. & a_{21} & [a_{22}] & a_{23} & \cdots & a_{2n} & \cdots \\ R_3 = & 0. & a_{31} & a_{32} & [a_{33}] & \cdots & a_{3n} & \cdots \\ & \vdots & & & & \ddots & \vdots & \\ R_n = & 0. & a_{n1} & a_{n2} & a_{n3} & \cdots & [a_{nn}] & \cdots \\ & \vdots & & & & & \vdots & \ddots \end{array}$$

Diagonalization Argument, cont

$$\begin{array}{rcl} R_1 = & 0. & [a_{11}] \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n} \quad \cdots \\ R_2 = & 0. & a_{21} \quad [a_{22}] \quad a_{23} \quad \cdots \quad a_{2n} \quad \cdots \\ R_3 = & 0. & a_{31} \quad a_{32} \quad [a_{33}] \quad \cdots \quad a_{3n} \quad \cdots \\ & \vdots & \\ R_n = & 0. & a_{n1} \quad a_{n2} \quad a_{n3} \quad \cdots \quad [a_{nn}] \quad \cdots \\ & \vdots & \qquad \qquad \qquad \qquad \qquad \qquad \vdots \quad \ddots \end{array}$$

- ▶ Now, we'll create a new real number R and show that it is not equal to any of the R_i 's in this sequence:
- ▶ Let $R = 0.a_1 a_2 a_3 \dots$ such that:

$$a_i = \begin{cases} 4 & d_{ii} \neq 4 \\ 5 & d_{ii} = 4 \end{cases}$$

- ▶ Clearly, this new number R differs from each number R_i in the table in at least one digit (its i 'th digit)

Diagonalization Argument, concluded

- ▶ Since R is not in the table, this is not a complete enumeration of all reals between 0 and 1
- ▶ Hence, the set of real between 0 and 1 is not countable
- ▶ Since the superset of any uncountable set is also uncountable, set of reals is uncountably infinite

