Connectivity in Graphs

A graph is connected if there is a path between every pair of vertices in the graph.

Example

Prove: Suppose graph $G$ has exactly two vertices of odd degree, say $u$ and $v$. Then $G$ contains a path from $u$ to $v$.

Example

Consider a graph with vertices \{x, y, z, w\} and edges $(x, y), (x, w), (x, z), (y, z)$

What are all the simple paths from $x$ to $w$?

What are all the simple paths from $x$ to $y$?

How many paths (can be non-simple) are there from $x$ to $y$?
### Circuits

- A **circuit** is a path that begins and ends in the same vertex.
- The paths `u, x, y, x` and `u, x, y, u` are both circuits.
- A **simple circuit** does not contain the same edge more than once.
- The path `u, x, y, x, u` is not a simple circuit, since it contains the same edge `x` twice.
- Length of a circuit is the number of edges it contains, for example, length of `u, x, y, u` is 3.
- In this class, we only consider circuits of length 3 or more.

### Cycles

- A **cycle** is a simple circuit with no repeated vertices other than the first and last ones.
- For instance, `u, x, a, b, x, y, u` is a circuit but not a cycle.
- However, `u, x, y, u` is a cycle.

### Example

- **Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.
- **Huh?** Recall that not every circuit is a cycle.
- According to this theorem, if we can find an odd length circuit, we can also find odd length cycle.
- **Example:** `d, c, a, b, c, d` is an odd length circuit, but graph also contains odd length cycle.

### Proof

- **Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.
  - **Proof by strong induction on the length of the circuit.**

### Proof, cont.

- **Prove:** If a graph has an odd length circuit, then it also has an odd length cycle.
Colorability and Cycles

**Prove:** If a graph is 2-colorable, then all cycles are of even length.

Example

Is this graph 2-colorable?

Distance Between Vertices

The distance between two vertices \( u \) and \( v \) is the length of the shortest path between \( u \) and \( v \).

What is the distance between \( u \) and \( b \)?

What is the distance between \( u \) and \( x \)?

What is the distance between \( x \) and \( w \)?

More Colorability and Cycles

**Prove:** If graph has no odd length cycles, then graph is 2-colorable.

The Algorithm

- Pick any vertex \( v \) in the graph.
- If a vertex \( u \) has odd distance from \( v \), color it **blue**
- Otherwise, color it **red**

Proof

- We will now prove: "If the graph does not have odd length cycles, the algorithm is correct."
- Correctness of the algorithm implies graph is 2-colorable.
- Proof by contradiction.
- Suppose graph does not have odd length cycles, but the algorithm produces an invalid coloring.
- Means there exist two vertices \( x \) and \( y \) that are assigned the same color.
Proof, cont.

Case 1: They are both assigned red

We know $n, m$ are both even

This means we now have an odd-length circuit involving $n, m$

By theorem from earlier, this implies that graph has odd length cycle, i.e., contradiction

Case 2 is exactly the same.

Putting It All Together

Theorem: A graph is 2-colorable if and only if it does not have odd-length cycles

Corollary: A graph is bipartite if and only if it does not have odd-length cycles

Example: Consider a graph $G$ with vertices $a, b, c, d, e, f$

Is $G$ partite if its edges are $(a, f), (e, f), (e, d), (c, d), (a, c)$?

Trees

A tree is a connected undirected graph with no cycles.

Examples and non-examples:

An undirected graph with no cycles is a forest.

Fact About Trees

Theorem: An undirected graph $G$ is a tree if and only if there is a unique simple path between any two of its vertices.

Leaves of a Tree

Given a tree, a vertex of degree 1 is called a leaf.

Important fact: Every tree with more than 2 nodes has at least two leaves.

Why is this true?
Number of Edges in a Tree

**Theorem:** A tree with $n$ vertices has $n - 1$ edges.

- **Proof is by induction on $n$**
  - **Base case:** $n = 1$ ✓
  - **Induction:** Assume property for tree with $n$ vertices, and show tree $T$ with $n + 1$ vertices has $n$ edges
    - Construct $T'$ by removing a leaf from $T$; $T'$ is a tree with $n$ vertices (tree because connected and no cycles)
    - By IH, $T'$ has $n - 1$ edges
    - Add leaf back: $n + 1$ vertices and $n$ edges

Rooted Trees

- A rooted tree has a designated root vertex and every edge is directed away from the root.
- Vertex $v$ is a **parent** of vertex $u$ if there is an edge from $v$ to $u$, and $u$ is called a **child** of $v$.
- Vertices with the same parent are called **siblings**.
- Vertex $v$ is an **ancestor** of $u$ if $v$ is $u$’s parent or an ancestor of $u$’s parent.
- Vertex $v$ is a **descendant** of $u$ if $u$ is $v$’s ancestor.

Questions about Rooted Trees

- Suppose that vertices $u$ and $v$ are siblings in a rooted tree.
- Which statements about $u$ and $v$ are true?
  1. They must have the same ancestors
  2. They can have a common descendant
  3. If $u$ is a leaf, then $v$ must also be a leaf

Subtrees

- Given a rooted tree and a node $v$, the **subtree** rooted at $v$ includes $v$ and its descendants.
- **Level** of vertex $v$ is the length of the path from the root to $v$.
- The **height** of a tree is the maximum level of its vertices.

$m$-ary Trees

- A rooted tree is called an $m$-ary tree if every vertex has no more than $m$ children.
- An $m$-ary tree where $m = 2$ is called a **binary tree**.
- A **full $m$-ary tree** is a tree where every internal node has exactly $m$ children.
- Which are full binary trees?

True-False Questions

1. Two siblings $u$ and $v$ must be at the same level.
2. A leaf vertex does not have a subtree.
3. The subtrees rooted at $u$ and $v$ can have the same height only if $u$ and $v$ are siblings.
4. The level of the root vertex is 1.
**Useful Theorem**

**Theorem:** An m-ary tree of height $h \geq 1$ contains at most $m^h$ leaves.

- Proof is by induction on height $h$.

**Corollary**

**Corollary:** If an m-ary tree has height $h$ and $n$ leaves, then $h \geq \lceil \log_m n \rceil$.

**Questions**

- What is maximum number of leaves in binary tree of height 5?
- If binary tree has 100 leaves, what is a lower bound on its height?
- If binary tree has 2 leaves, what is an upper bound on its height?

**Balanced Trees**

- An m-ary tree is balanced if all leaves are at levels $h$ or $h - 1$.
- "Every full tree must be balanced." – true or false?
- "Every balanced tree must be full." – true or false?