CS311H: Discrete Mathematics

Graph Theory II

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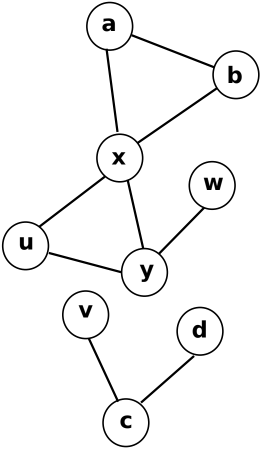
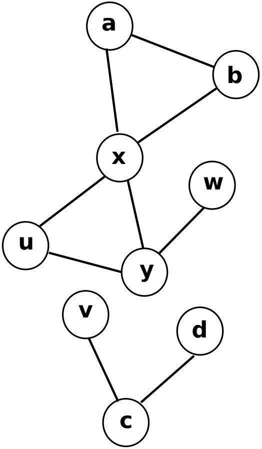
Connectivity in Graphs



- Typical question: Is it possible to get from some node u to another node v?
- Example: Train network – if there is path from u to v, possible to take train from u to v and vice versa.
- If it's possible to get from u to v, we say u and v are connected and there is a path between u and v

Paths

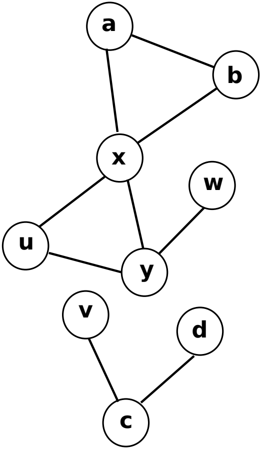


- A path between u and v is a sequence of edges that starts at vertex u, moves along adjacent edges, and ends in v.
- Example: u, x, y, w is a path, but u, y, v and u, a, x are not
- Length of a path is the number of edges traversed, e.g., length of u, x, y, w is 3
- A simple path is a path that does not repeat any edges
- u, x, y, w is a simple path but u, x, u is not

Example

- Consider a graph with vertices $\{x, y, z, w\}$ and edges $(x, y), (x, w), (x, z), (y, z)$
- What are all the simple paths from z to w?
- What are all the simple paths from x to y?
- How many paths (can be non-simple) are there from x to y?

Connectedness



- A graph is connected if there is a path between every pair of vertices in the graph
- Example: This graph not connected; e.g., no path from x to d
- A connected component of a graph G is a maximal connected subgraph of G

Example

- Prove: Suppose graph G has exactly two vertices of odd degree, say u and v. Then G contains a path from u to v.
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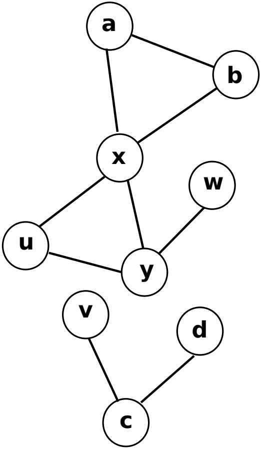
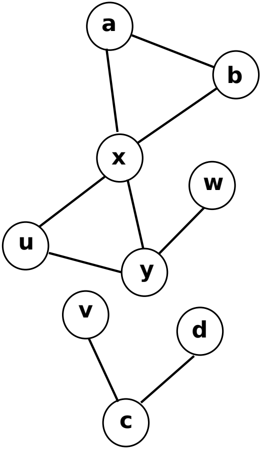
Circuits



- A circuit is a path that begins and ends in the same vertex.
- u, x, y, x, u and u, x, y, u are both circuits
- A simple circuit does not contain the same edge more than once
- u, x, y, u is a simple circuit, but u, x, y, x, u is not
- Length of a circuit is the number of edges it contains, e.g., length of u, x, y, u is 3
- In this class, we only consider circuits of length 3 or more

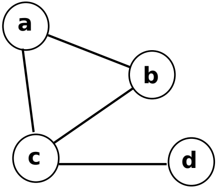
Cycles



- A cycle is a simple circuit with no repeated vertices other than the first and last ones.
- For instance, u, x, a, b, x, y, u is a circuit but not a cycle
- However, u, x, y, u is a cycle

Example

- Prove: If a graph has an odd length circuit, then it also has an odd length cycle.
- Huh? Recall that not every circuit is a a cycle.
- According to this theorem, if we can find an odd length circuit, we can also find odd length cycle.
- Example: d, c, a, b, c, d is an odd length circuit, but graph also contains odd length cycle



Proof

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

- Proof by strong induction on the length of the circuit.
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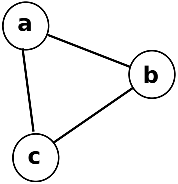
- 

Proof, cont.

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

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Proof, cont.

Prove: If a graph has an odd length circuit, then it also has an odd length cycle.

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Colorability and Cycles

Prove: If a graph is 2-colorable, then all cycles are of even length.

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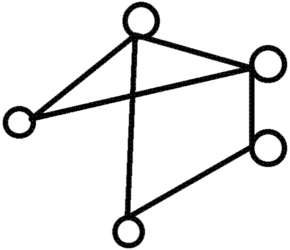
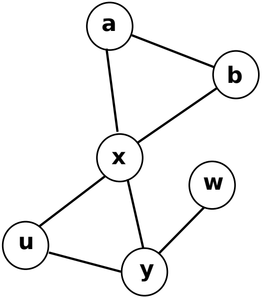
Example



- Is this graph 2-colorable?

Distance Between Vertices



- The distance between two vertices u and v is the length of the shortest path between u and v
- What is the distance between u and b?
- What is the distance between u and x?
- What is the distance between x and w?

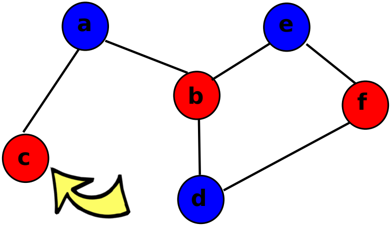
More Colorability and Cycles

Prove: If graph has no odd length cycles, then graph is 2-colorable.

- To prove this, we first consider an algorithm for coloring the graph with two colors.
- Then, we will show that this algorithm works if graph does not have odd length cycles.

The Algorithm

- Pick any vertex v in the graph.
- If a vertex u has odd distance from v, color it blue
- Otherwise, color it red

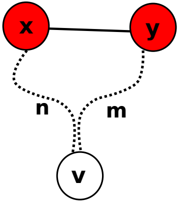


Proof

- We will now prove: "If the graph does not have odd length cycles, the algorithm is correct."
- Correctness of the algorithm implies graph is 2-colorable.
- Proof by contradiction.
- Suppose graph does not have odd length cycles, but the algorithm produces an invalid coloring.
- Means there exist two vertices x and y that are assigned the same color.

Proof, cont.

- Case 1: They are both assigned red



- We know n, m are both even
- This means we now have an odd-length circuit involving n, m
- By theorem from earlier, this implies that graph has odd length cycle, i.e., contradiction
- Case 2 is exactly the same.

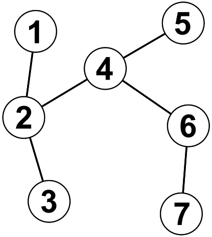
Putting It All Together

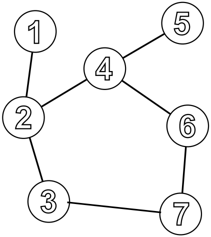
- Theorem: A graph is 2-colorable if and only if it does not have odd-length cycles
- Corollary: A graph is bipartite if and only if it does not have odd-length cycles
- Example: Consider a graph G with vertices a, b, c, d, e, f
  - Is G partitle if its edges are $(a, f), (e, f), (e, d), (c, d), (a, c)$?

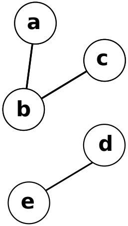
Trees

- A tree is a connected undirected graph with no cycles.

- Examples and non-examples:







- An undirected graph with no cycles is a forest.

Fact About Trees

Theorem: An undirected graph G is a tree if and only if there is a unique simple path between any two of its vertices.

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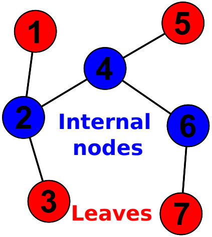
Leaves of a Tree

- Given a tree, a vertex of degree 1 is called a leaf.



- **Important fact: Every tree with more than 2 nodes has at least two leaves.**

Why is this true?

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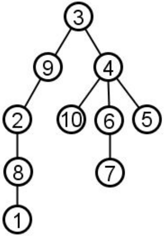
Number of Edges in a Tree

Theorem: A tree with n vertices has $n - 1$ edges.

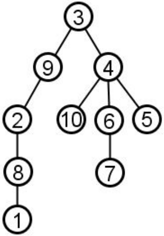
- Proof is by induction on n
- Base case: $n = 1$ ✓
- Induction: Assume property for tree with n vertices, and show tree T with $n + 1$ verices has n edges
- Construct T' by removing a leaf from T; T' is a tree with n vertices (tree because connected and no cycles)
- By IH, T' has $n - 1$ edges
- Add leaf back: $n + 1$ vertices and n edges

Rooted Trees



- A rooted tree has a designated root vertex and every edge is directed away from the root.
- Vertex v is a parent of vertex u if there is an edge from v to u; and u is called a child of v
- Vertices with the same parent are called siblings
- Vertex v is an ancestor of u if v is u's parent or an ancestor of u's parent.
- Vertex v is a descendant of u if u is v's ancestor

Questions about Rooted Trees

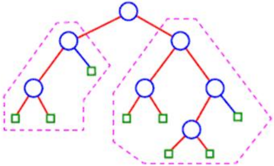


- Suppose that vertices u and v are siblings in a rooted tree.
- Which statements about u and v are true?

1. They must have the same ancestors
2. They can have a common descendant
3. If u is a leaf, then v must also be a leaf

Subtrees

- Given a rooted tree and a node v, the subtree rooted at v includes v and its descendants.



- Level of vertex v is the length of the path from the root to v.

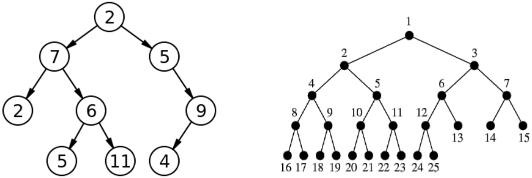
- The height of a tree is the maximum level of its vertices.

True-False Questions

1. Two siblings u and v must be at the same level.
2. A leaf vertex does hot have a subtree.
3. The subtrees rooted at u and v can have the same height only if u and v are siblings.
4. The level of the root vertex is 1.

m-ary Trees

- A rooted tree is called an m-ary tree if every vertex has no more than m children.
- An m-ary tree where $m = 2$ is called a binary tree.
- A full m-ary tree is a tree where every internal node has exactly m children.
- Which are full binary trees?



Useful Theorem

Theorem: An m-ary tree of height $h \geq 1$ contains at most m^h leaves.

- Proof is by induction on height h.
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Corollary

Corollary: If m-ary tree has height h and n leaves, then $h \geq \lceil log_m n \rceil$

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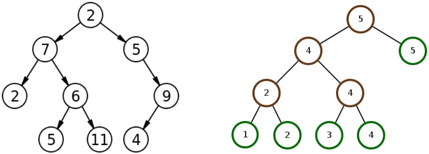
Questions

- What is maximum number of leaves in binary tree of height 5?
- If binary tree has 100 leaves, what is a lower bound on its height?
- If binary tree has 2 leaves, what is an upper bound on its height?

Balanced Trees

- An m-ary tree is balanced if all leaves are at levels h or $h-1$



- "Every full tree must be balanced." – true or false?
- "Every balanced tree must be full." – true or false?