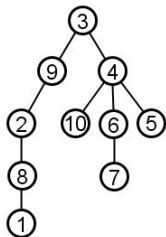


CS311H: Discrete Mathematics

Graph Theory III

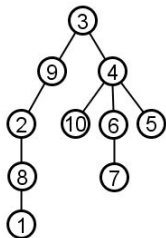
Instructor: Işıl Dillig

Rooted Trees



- ▶ A **rooted tree** has a designated root vertex and every edge is directed away from the root.
- ▶ Vertex v is a **parent** of vertex u if there is an edge from v to u ; and u is called a **child** of v
- ▶ Vertices with the same parent are called **siblings**
- ▶ Vertex v is an **ancestor** of u if v is u 's parent or an ancestor of u 's parent.
- ▶ Vertex v is a **descendant** of u if u is v 's ancestor

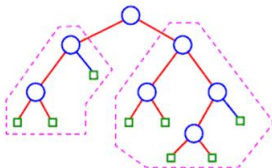
Questions about Rooted Trees



- ▶ Suppose that vertices u and v are siblings in a rooted tree.
- ▶ Which statements about u and v are true?
 1. They must have the same ancestors
 2. They can have a common descendant
 3. If u is a leaf, then v must also be a leaf

Subtrees

- ▶ Given a rooted tree and a node v , the **subtree** rooted at v includes v and its descendants.



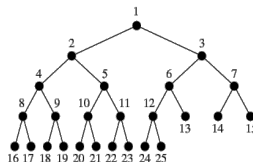
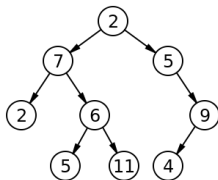
- ▶ **Level** of vertex v is the length of the path from the root to v .
- ▶ The **height** of a tree is the maximum level of its vertices.

True-False Questions

1. Two siblings u and v must be at the same level.
2. A leaf vertex does not have a subtree.
3. The subtrees rooted at u and v can have the same height only if u and v are siblings.
4. The level of the root vertex is 1.

m -ary Trees

- ▶ A rooted tree is called an **m -ary tree** if every vertex has no more than m children.
- ▶ An m -ary tree where $m = 2$ is called a **binary tree**.
- ▶ A **full m -ary tree** is a tree where every internal node has exactly m children.
- ▶ Which are full binary trees?



Useful Theorem

Theorem: An m -ary tree of height $h \geq 1$ contains at most m^h leaves.

- ▶ Proof is by strong induction on height h .



Corollary

Corollary: If m -ary tree has height h and n leaves, then $h \geq \lceil \log_m n \rceil$

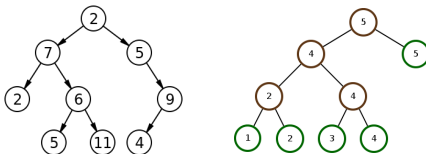


Questions

- ▶ What is maximum number of leaves in binary tree of height 5?
- ▶ If binary tree has 100 leaves, what is a lower bound on its height?
- ▶ If binary tree has 2 leaves, what is an upper bound on its height?

Balanced Trees

- ▶ An m -ary tree is balanced if all leaves are at levels h or $h - 1$



- ▶ "Every full tree must be balanced." – true or false?
- ▶ "Every balanced tree must be full." – true or false?

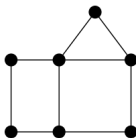
Theorem about Full and Balanced Trees

Theorem: For a full and balanced m -ary tree with height h and n leaves, we have $h = \lceil \log_m n \rceil$

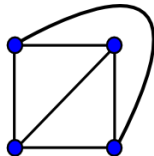
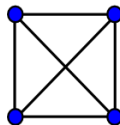
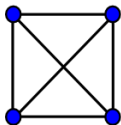


Planar Graphs

- ▶ A graph is called **planar** if it can be drawn in the plane without any edges crossing (called **planar representation**).



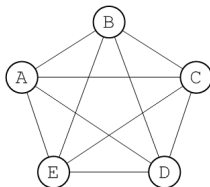
- ▶ Is this graph planar?



- ▶ In this class, we will assume that every planar graph has at least 3 edges.

A Non-planar Graph

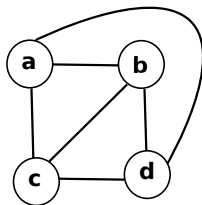
- ▶ The complete graph K_5 is not planar:



- ▶ Why can K_5 not be drawn without any edges crossing?

Regions of a Planar Graph

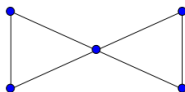
- ▶ The planar representation of a graph splits the plane into **regions** (sometimes also called **faces**):



- ▶ **Degree of a region** R , written $\deg(R)$, is the number of edges bordering R
- ▶ Here, all regions have degree 3.

Examples

- ▶ How many regions does this graph have?

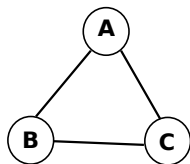
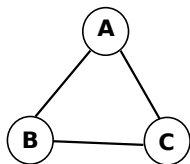


- ▶ What is the degree of its outer region?
- ▶ How many regions does a graph have if it has no cycles?
- ▶ Given a planar **simple** graph with at least 3 edges, what is the minimum degree a region can have?
- ▶ What is the relationship between $\sum \deg(R)$ and the number of edges?

Euler's Formula

Euler's Formula: Let $G = (V, E)$ be a planar connected graph with regions R . Then, the following formula always holds:

$$|R| = |E| - |V| + 2$$



All planar representations of a graph split the plane into the same number of regions!

Proof of Euler's Formula

- ▶ **Case 1:** G does not have cycles (i.e., a tree)
- ▶ If G has $|V|$ nodes, how many edges does it have?
- ▶ How many regions does it have?
- ▶ $|R| = 1 = (|V| - 1) - |V| + 2 \quad \checkmark$

Proof, cont.

- ▶ **Case 2:** G has at least one cycle.
- ▶ The proof is by induction on the number of edges.
- ▶ **Base case:** G has 3 edges (i.e., a triangle)
- ▶ **Induction:** Suppose Euler's formula holds for planar connected graphs with e edges and at least one cycle.
- ▶ We need to show it also holds for planar connected graphs with $e + 1$ edges and at least one cycle.

Proof, cont.

- ▶ Create G' by removing one edge from the cycle $\Rightarrow$ has e edges
- ▶ If G' doesn't have cycles, we know $|R| = e - |V| + 2$ (case 1)
- ▶ If G' has cycles, we know from IH that $|R| = e - |V| + 2$
- ▶ Now, add edge back in; G has $e + 1$ edges and $|V|$ vertices
- ▶ How many regions does G have? $|R| + 1$
- ▶ $e + 1 - |V| + 2 = |R| + 1$ ✓

An Application of Euler's Formula

- ▶ Suppose a connected planar simple graph G has 6 vertices, each with degree 4.
- ▶ How many regions does a planar representation of G have?
- ▶ How many edges?
- ▶ How many regions?

A Corollary of Euler's Formula

Theorem: Let G be a connected planar simple graph with v vertices and e edges. Then $e \leq 3v - 6$

- ▶ **Proof:** Suppose G has r regions.
- ▶ Recall: $2e = \sum \deg(R)$
- ▶ Hence, $2e \geq 3r$
- ▶ From Euler's formula, $3r = 3e - 3v + 6$; thus
 $2e \geq 3e - 3v + 6$
- ▶ Implies $e \leq 3v - 6$ ✓

Why is this Theorem Useful?

Theorem: Let G be a connected planar simple graph with v vertices and e edges. Then $e \leq 3v - 6$

- ▶ Can be used to show graph is not planar.
- ▶ **Example:** Prove that K_5 is not planar.
- ▶ How many edges does K_5 have?
- ▶ $3 \cdot 5 - 6 = 9$, but $10 \not\leq 9$

Another Corollary

Theorem: If G is a connected, planar simple graph, then it has a vertex of degree not exceeding 5.

- ▶ Proof by contradiction: Suppose every vertex had degree at least 6
- ▶ What lower bound does this imply on number of edges?
- ▶