# CS311H: Discrete Mathematics

## Recurrence Relations

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# Recall: Recursively Defined Sequences

- In previous lectures, we looked at recursively-defined sequences
- Example: What sequence is this?

$$\begin{aligned} a_0 &= 1 \\ a_n &= a_{n-1} + 1 \end{aligned}$$

- Another example: Fibonacci numbers

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \end{aligned}$$

# Recurrence Relations

- Recurively defined sequences are often referred to as recurrence relations
- The base cases in the recursive definition are called initial values of the recurrence relation
- Example: Write recurrence relation representing number of bacteria in $n$'th hour if colony starts with 5 bacteria and doubles every hour?

# Closed Form Solutions

- Recurrence relations are often very natural to define, but we usually need to find closed form solution
- Recall: Closed form solution defines $n$'th number in the sequence as a function of $n$
- What is closed form solution to the following recurrence?

$$
\begin{aligned}
a_0 &= 0 \\
a_n &= a_{n-1} + n
\end{aligned}
$$

# Closed Form Solutions of Recurrence Relations

- Given an arbitrary recurrence relation, is there a mechanical way to obtain the closed form solution?
- Not for arbitrary, but for a subclass of recurrence relations
- A linear homogeneous recurrence relation with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

  where each $c_i$ is a constant and $c_k$ is non-zero
- The value of $k$ is called the degree of the recurrence relation
- The recurrence relation for Fibonacci numbers is a degree $2$ linear homogeneous recurrence

# Examples and Non-Examples

- Which of these are linear homogenous recurrence relations with constant coefficients?
  - $a_n = a_{n-1} + 2a_{n-5}$
  - $a_n = 2a_{n-2} + 5$
  - $a_n = a_{n-1} + n$
  - $a_n = a_{n-1} \cdot a_{n-2}$
  - $a_n = n \cdot a_{n-1}$

# Characteristic Polynomial

- Cook-book recipe for solving linear homogenous recurrence relations with constant coefficients
- Definition: The characteristic equation of a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$ is

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k$$

- i.e., Replace $a_{n-i}$ with $r^{n-i-(n-k)}$

# Characteristic Equation Examples

- What are the characteristic equations for the following recurrence relations?
  - $f_n = f_{n-1} + f_{n-2}$
  - $a_n = 2a_{n-1}$
  - $a_n = 2a_{n-1} + 5a_{n-3}$

# Characteristic Roots

- The characteristic roots of a linear homogeneous recurrence relation are the roots of its characteristic equation.
- What are the characteristic roots of the following recurrence relations?
  - $a_n = 2a_{n-1} + 3a_{n-2}$
  - $f_n = f_{n-1} + f_{n-2}$

# Theorem I for Solving Linear Homogenous Recurrence Relations

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with $k$ distinct characteristic roots $r_1, \ldots, r_k$.

- Then the closed form solution for $a_n$ is of the form:

$$\alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

- Furthermore, given $k$ initial conditions, the constants $\alpha_1, \ldots, \alpha_k$ are uniquely determined
- Note: Won't do the proof because requires a good amount of linear algebra

# Example

Find a closed form solution for the recurrence $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 7$

- Characteristic equation:
- Characteristic roots:
- Coefficients:
- Closed-form solution:

# Generalized Theorem

- So far, we assume all characteristic roots are distinct – what happens if this is not the case?
- Theorem: Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a recurrence relation with $t$ distinct characteristic roots $r_1, \ldots, r_k$ with multiplicities $m_1, \ldots, m_k$. Then solutions are of the form:

$$a_n = \sum_{i=0}^{t} (\alpha_{i,0} + \alpha_{i,1} \cdot n + \ldots + \alpha_{i,m_{i-1}} \cdot n^{m_i - 1}) r_i^n$$

# An Example

- Find closed form of recurrence $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ with initial conditions $a_0 = 1, a_1 = 3, a_2 = 7$
- Characteristic equation:
- Characteristic roots:
- Solution form:
- Coefficients:

# Solving Linear Non-Homogeneous Recurrence Relations

- How do we solve linear, but non-homogeneous recurrence relations, such as $a_n = 2a_{n-1} + 1$?
- A linear non-homogeneous recurrence relation with constant coefficients is of the form:

$$a_n = c_1 a_{n-1} + a_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

- The recurrence obtained by dropping $F(n)$ is called the associated homogeneous recurrence relation

# Particular Solution

- A particular solution for a recurrence relation is one that satisfies the recurrence but not necessarily the initial conditions
- Example: Consider the recurrence $a_n = a_{n-1} + 1$ with initial condition $a_0 = 5$
- A particular solution for this recurrence is $a_n = n$, but it does not satisfy the initial condition

# Theorem about Linear Non-homogeneous Recurrences

Suppose $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n)$ has particular solution $a_n^p$, and $a_n^h$ is solution for associated homogeneous recurrence. Then every solution is of the form $a_n^p + a_n^h$.

# Why is this theorem useful?

- If we can find a particular solution, then we can also mechanically find a solution that satisfies initial conditions.
- Example: Solve the recurrence relation $a_n = 3a_{n-1} + 2n$ with initial condition $a_1 = 3$
- A particular solution: $-n - \frac{3}{2}$ (Why?)
- Solutions for homogeneous recurrence:
- Solutions for recurrence:
- Solve for $\alpha$:

# How do we find a particular solution?

Theorem: Consider $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n)$ where:

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$$

- Case 1: If $s$ is not a root of the associated characteristic equation, then there exists a particular solution of the form:

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

- Case 2: If $s$ is a root with multiplicity $m$ of the characteristic equation, then there exists a solution of the form:

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

# Example I

- Consider again the recurrence $a_n = 3a_{n-1} + 2n$
- Here, $s = 1$ and characteristic root is $3$
- Hence, there exists a particular solution of the form $p_1 n + p_0$
- Now, solve for $p_0, p_1$:

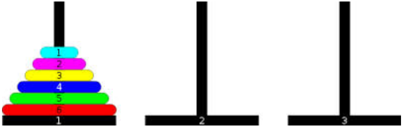
$$p_1 n + p_0 = 3(p_1(n-1) + p_0) + 2n$$

- Rearrange: $2n(p_1 + 1) + (2p_0 - 3p_1) = 0$
- A solution $p_1 = -1, p_0 = -\frac{3}{2}$
- A particular solution: $-n - \frac{3}{2}$

# Example II

- Find a particular solution for $a_n = 6a_{n-1} - 9a_{n-2} + 2^n$
- Characteristic root:
- Particular solution of the form:
- Find $p_0$ such that $p_0 \cdot 2^n = 6(p_0 \cdot 2^{n-1}) - 9(p_0 \cdot 2^{n-2}) + 2^n$
- Solve for $p_0$:
- Particular solution:

# Towers of Hanoi




- Given 3 pegs where first peg contains $n$ disks
- Goal: Move all the disks to a different peg (e.g., second one)
- Rule 1: Larger disks cannot rest on top of smaller disks
- Rule 2: Can only move the top-most disk at a time
- Question: How many steps does it take to move all $n$ disks?

# A Recursive Solution

- Solve recursively – $T_n$ is number of steps to move $n$ disks
- Base case: $n = 1$, move disk from first peg to second: $T_1 = 1$
- Induction: Suppose we can move $n - 1$ disks in $T_{n-1}$ steps; how many steps does it take to move $T_n$ disks?
- Idea: First move the topmost $n - 1$ disks to peg 3; can be done in $T_{n-1}$ steps
- Now, move bottom-most disk to peg 2 – takes just 1 step
- Finally, recursively move $n - 1$ disks in peg 3 to peg 2 – can be done in $T_{n-1}$ steps

# Towers of Hanoi, cont.

- Recurrence relation:
- Initial condition:
- Now find closed form for $T_n$
- What is a particular solution?
- Solution for homogeneous recurrence:
- Solve for $\alpha$:
- Solution for recurrence: