CS389L: Automated Logical Reasoning

Lecture 12: Decision Procedure for the Theory of Rationals

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Overview

- Today: Talk about how to decide satisfiability of the quantifier-free fragment of $T_{\mathbb{Q}}$
- We'll only consider quantifier free conjunctive $T_{\mathbb{Q}}$ formulas (i.e., no disjunctions)
- Most common technique for deciding satisfiability in $T_{\mathbb{Q}}$ is Simplex algorithm
- Simplex algorithm developed by Dantzig in 1949 for solving linear programming problems
- Since deciding satisfiability of qff conjunctive formulas is a special case of linear programming, we can use Simplex

The Plan

- Overview of linear programming
- Satisfiability as linear programming
- Simplex algorithm

Linear Programming

- In a linear programming (LP) problem, we have an $m \times n$ matrix A, an m-dimensional vector $\vec{b}$, and n-dimensional vector $\vec{c}$
- Want to find a solution for $\vec{x}$ maximizing objective function

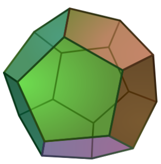
$$\vec{c}^T\vec{x}$$

subject to linear inequality constraint

$$A\vec{x} \leq \vec{b}$$

- Very important problem; applications in airline scheduling, transportation, telecommunications, finance, production management, marketing, networking, compilers ...

Geometric Formulation



- For $m \times n$ matrix A, the system $A\vec{x} \leq \vec{b}$ forms a convex polytope in n-dimensional space
- Polytope is generalization of polyhedron from 3-dim space to higher dimensional space
- Convexity: For all pairs of points $\vec{v_1}, \vec{v_2}$ and for any $\lambda \in [0,1]$, the point $\lambda\vec{v_1} + (1-\lambda)\vec{v_2}$ also lies in polytope
- Goal of linear programming: Find a point that (i) lies inside the polytope, and (ii) maximizes the value of $\vec{c}^T\vec{x}$

Linear Programming Lingo

- In LP, a value of $\vec{x}$ that satisfies constraints $A\vec{x} \leq \vec{b}$ called feasible solution; otherwise, called infeasible solution
- Example: Maximize $2y - x$ subject to:

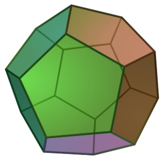
$$\begin{array}{rcl} x + y & \leq & 3 \\ 2x - y & \leq & -5 \end{array}$$

- Is $(0,0)$ a feasible solution?
- What about $(-2,1)$?
- For a given solution for $\vec{x}$, the corresponding value of objective function $\vec{c}^T\vec{x}$ called objective value
- What is objective value for $(-2,1)$?

Linear Programaming Lingo, cont

- A feasible solution whose objective value is maximum over all feasible solutions called optimal solution
- If a linear program has no feasible solutions, the linear program is infeasible
- If optimal solution is ∞, then problem is called unbounded

Geometric Interpretation



- Feasible solution is a point within the polytope
- The linear programming problem is infeasible if the polytope defined by $A\vec{x} \leq \vec{b}$ is empty
- An LP problem is unbounded if the polytope is open in the direction of the objective function
- Question: If polytope is not closed, does this mean optimal solution is ∞?
- Since the polytope defined by $A\vec{x} \leq \vec{b}$ is convex, the optimal solution for bounded LP problem must lie on exterior boundary of polytope

Deciding $T_{\mathbb{Q}}$ as Linear Program

- How do we determine $T_{\mathbb{Q}}$ satisfiability using LP?
- First, convert $T_{\mathbb{Q}}$ formula to NNF.
- In this form, every atomic formula is of the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \bowtie c \quad (\bowtie \in \{=, \neq, \geq, <\})$$

- First, rewrite it as equisat formula containing only $\leq$ and > 0

$$\begin{array}{lll} \vec{a}^T\vec{x} \geq c & \Rightarrow & \\ \vec{a}^T\vec{x} < c & \Rightarrow & \\ \vec{a}^T\vec{x} = c & \Rightarrow & \\ \vec{a}^T\vec{x} \neq c & \Rightarrow & (\vec{a}^T\vec{x} + y \leq c \land y > 0) \lor \\ & & (-\vec{a}^T\vec{x} + y \leq -c \land y > 0) \end{array}$$

Deciding $T_{\mathbb{Q}}$ as Linear Program, cont

- Current formula in NNF and no negations
- Each atomic formula is one of three forms:
  1. $a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i$
  2. $\alpha_{i1}x_1 + \ldots + \alpha_{in}x_n + y \leq \beta_i$
  3. $y > 0$
- Next, convert to DNF: Formula is satisfiable iff any of the clauses satisfiable
- Thus, want to formulate each clause as a linear program

Deciding $T_{\mathbb{Q}}$ as Linear Program, cont

- Each clause is of the following form:

$$\begin{array}{ll} & \bigwedge a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i \\ \land & \bigwedge \alpha_{i1}x_1 + \ldots + \alpha_{in}x_n + y \leq \beta_i \\ \land & y > 0 \end{array}$$

- How can we decide whether this constraint is satisfiable by formulating it as an LP problem?
- This constraint is satisfiable iff the optimal solution of the following LP problem is strictly positive:

  Maximize y
  Subject to:
  $\bigwedge a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i \land \bigwedge \alpha_{i1}x_1 + \ldots + \alpha_{in}x_n + y \leq \beta_i$
- Why?

Satisfiability as Linear Programming

- Thus, we can formulate satisfiability of every qff conjunctive $T_{\mathbb{Q}}$ formula as a linear programming problem.
- Three popular methods for solving LP problems:
  1. Ellipsoid method (Khachian, 1979)
  2. Interior-point algorithm (Karmarkar, 1984)
  3. Simplex algorithm (Dantzig, 1949)
- Among these, ellipsoid and interior-point method are polynomial-time, but Simplex is worst-case exponential
- Despite this, Simplex remains most popular and performs better for most problems of interest

Prerequisites for Simplex

- To apply Simplex, we have to transform linear inequality system into standard form and then into slack form
- Standard form:

  Maximize $\vec{c}^T\vec{x}$
  Subject to: $A\vec{x} \leq \vec{b}$
  $\vec{x} \geq 0$

- Bad news: In general, not all problems require non-negative solution, thus $\vec{x} \leq 0$ requirement unrealistic
- Good news: We can convert every LP problem into an equisatisfiable standard form representation
- Equisat. means original problem has optimal objective value c iff problem in standard form has optimal objective value c

Conversion to Standard Form

- Main idea: Any negative variable can be written as difference of two non-negative integers
- For each such variable, introduce two new variables x_i' and x_i''
- Add non-negativity constraints: $x_i' \geq 0$ and $x_i'' \geq 0$
- Express x_i as $x_i' - x_i''$ by substituting $x_i' - x_i''$ for each occurence of x_i

Standard Form Example

- Consider the following linear program:

  Maximize $2x_1 - 3x_2$
  Subject to:
  $$x_1 + x_2 \leq 7$$
  $$-x_1 - x_2 \leq -7$$
  $$x_1 - 2x_2 \leq 4$$
  $$x_1 \geq 0$$

- Variable x_2 does not have non-negativity constraint; thus rewrite it as $x_2' - x_2''$
- Equisatisfiable system in standard form:

  Maximize $2x_1 - 3x_2' + 3x_2''$
  Subject to:
  $$x_1 + x_2' - x_2'' \leq 7$$
  $$-x_1 - x_2' + x_2'' \leq -7$$
  $$x_1 - 2x_2' + 2x_2'' \leq 4$$
  $$x_1, x_2', x_2'' \geq 0$$

Conversion to Slack Form

- To apply Simplex, we need inequalities to be in slack form
- In slack form, we only have equalities; the only inequality allowed is non-negativity constraints
- For each inequality $A_i\vec{x} \leq b_i$, introduce a new slack variable s_i
- Slack variables measure the difference (i.e., "slack") between left-hand and right-hand side
- Rewrite inequality as equality $s_i = b_i - A_i x$ and introduce non-negativity constraint $s_i \geq 0$

Slack Form Conversion Example

- Consider LP problem from previous example:

  Maximize $2x_1 - 3x_2 + 3x_3$
  Subject to:
  $$x_1 + x_2 - x_3 \leq 7$$
  $$-x_1 - x_2 + x_3 \leq -7$$
  $$x_1 - 2x_2 + 2x_3 \leq 4$$
  $$x_1, x_2, x_3 \geq 0$$

- In slack form:

  Maximize $2x_1 - 3x_2 + 3x_3$
  Subject to:
  $$x_4 = 7 - x_1 - x_2 + x_3$$
  $$x_5 = -7 + x_1 + x_2 - x_3$$
  $$x_6 = 4 - x_1 + 2x_2 - 2x_3$$
  $$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Basic and Non-Basic Variables

- In slack form, there is exactly one variable on the left hand side of equalities
- Variables appearing on the left-hand side called basic variables
- Variables appearing on RHS called non-basic variables
- Invariant: Only non-basic variables can appear in the objective function
- Initially, all basic variables are slack variables, but this will change as algorithm proceeds

Slack Form: Summary

- We'll denote the set of basic variables by B and non-basic variables by N.
- Then we'll write the slack form as a set of equations of the following form:

$$z = v + \sum_{x_j \in N} c_j x_j \quad \text{(objective function)}$$

$$x_i = b_i - \sum_{x_j \in N} a_{ij} x_j \quad (\text{for every } x_i \in B)$$

- There are implicit non-negativity constraints on all variables, but we omit them
- Question: Given original matrix A is $m \times n$, what is $|B|$?

Basic Solution

- For each LP problem in slack form, there is a basic solution
- To obtain basic solution, set all non-basic variables to zero
- Compute values of basic variables on the left-hand side
- What is basic solution for this slack form?

$$\begin{aligned} z &= 3x_1 + x_2 + 2x_3 \\ x_4 &= 30 - x_1 - x_2 - 3x_3 \\ x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\ x_6 &= 36 - 4x_1 - x_2 - 2x_3 \end{aligned}$$

- Basic solution called feasible basic solution if it doesn't violate non-negativity constraints

Simplex Algorithm Phases

- Simplex algorithm has two phases:
  1. Phase I: Compute a feasible basic solution, if one exists
  2. Phase II: Optimize value of objective function
- Understanding Phase I relies on understanding phase II
- Thus, we'll talk about Phase II first

Simplex Algorithm Optimization Phase Overview

- Starting with a feasible basic solution, each iteration rewrites one slack form into an equivalent slack form
- This rewriting is similar to Gaussian elimination: involves pivot operations on matrix
- Geometrically, each iteration of Simplex "walks" from one vertex to an adjacent vertex until it reaches a local maximum
- By convexity, local optimum is global optimum; thus algorithm can safely stop when local maximum is reached

Simplex Algorithm Optimization Phase

- When rewriting one slack form to another, goal is to increase value of objective function associated with basic solution
- Recall: Objective function is $z = v + \sum_{x_j \in N} c_j x_j$
- How can we increase value of z?
- If there is a term $c_j x_j$ with positive c_j, we can increase value of z by increasing x_j's value, i.e., by making x_j a basic variable
- What if there are no positive c_j's?
- Then, we know we can't increase value of z, thus we are done!

Simplex Algorithm Optimization Phase, cont

- Suppose we can increase objective value, i.e., there exists a term $c_j x_j$ with positive c_j
- We want to increase x_j's value, but is there a limit on how much we can increase x_j?
- Consider equality $x_i = b_i - a_{ij} x_j - \ldots$
- Observe: If a_{ij} is positive and we increase x_j beyond $\frac{b_i}{a_{ij}}$, x_i becomes negative and we violate constraints
- Thus, the amount by which we can increase x_j is limited by the smallest $\frac{b_i}{a_{ij}}$ among all i's
- If there is no positive coefficient a_{ij}, we can increase x_j (and thus z) without limit $\Rightarrow$ optimal solution $= \infty$

Summary

- Thus, given term $c_j x_j$ with positive c_j in objective function, we want to increase x_j as much as possible
- To increase x_j as much as possible, we find equality that most severely restricts how much we can increase x_j
- Equality that most severaly restricts x_j has following characteristics:
  1. x_j's coefficient a_{ij} is positive (otherwise doesn't limit x_j)
  2. has smallest value of $\frac{b_i}{a_{ij}}$ (most severely restricting)

Simplex Algorithm Optimization Phase, cont

- Suppose equality with basic var. x_i is most restrictive for x_j
- Swap roles of x_i and x_j by making x_j basic and x_i non-basic
- To do this, rewrite x_j in terms of x_i and plug this in to all other equations; this operation is called a pivot
- After performing this pivot operation, what is new value of x_j?
- We have increased the value of x_j from 0 to $\frac{b_i}{a_{ij}}$
- Thus, after performing pivot we still have feasible solution but objective value is now greater

Simplex Optimization Phase Summary

- Pivot operation exchanges a basic variable with a non-basic variable to increase objective value of basic solution
- Simplex repeats this pivot operation until one of two conditions hold:
  1. All coefficients in objective function are negative $\Rightarrow$ optimal solution found
  2. There exists a non-basic variable x_j with positive coefficient c_j in objective function, but all coefficients a_{ij} are negative $\Rightarrow$ optimal solution $= \infty$

Example

$$\begin{aligned} z &= 3x_1 + x_2 + 2x_3 \\ x_4 &= 30 - x_1 - x_2 - 3x_3 \\ x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\ x_6 &= 36 - 4x_1 - x_2 - 2x_3 \end{aligned}$$

- How can we increase value of objective function?
-
- Which equality restricts x_1 the most?
- Rewrite x_1 in terms of x_6:

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

Example, cont

- Plug this in for x_1 in all other equations (i.e., pivot):

$$\begin{aligned} z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\ x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{4} - \frac{x_6}{4} \\ x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\ x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \end{aligned}$$

- How can we increase value of z?
-
- Which equality restricts x_3 the most?
- What is x_3 in terms of x_5, x_2, x_6?

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

Example, cont

- New slack form after making x_3 basic, x_5 non-basic:

$$\begin{aligned} z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\ x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\ x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\ x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \end{aligned}$$

- Can we increase z?
- Which equality restricts x_2 the most?
-
- Solve x_2 in terms of x_3:

$$x_2 = 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6$$

Example, cont.

- New slack form after making x_2 basic, x_3 non-basic:

$$\begin{array}{lcl} z & = & 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\ x_1 & = & 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\ x_2 & = & 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\ x_4 & = & 18 - \frac{x_3}{2} + \frac{x_5}{2} \end{array}$$

- Can we increase objective value?
- What is optimal objective value?
- What is optimal solution?

Degenerate Problems

- Can the objective value decrease between two successive iterations?
- Objective value can't decrease; but can it stay the same? Yes
- Example: Suppose we make x_2 the new basic variable, and most constraining equality is:

$$x_1 = x_2 + 2x_3 + x_4$$

- x_2's old value was 0; what is its new value? Also 0
- These kinds of problems where objective value can stay the same after pivoting are called degenerate problems

Degenerate Problems and Termination

- If problem is not degenerate, Simplex guaranteed to terminate for any pivot selection strategy (b/c objective value increases)
- Bad news: For degenerate problems, Simplex might not terminate
- Good news: There are pivot selection strategies for which Simplex is always guaranteed to terminate, even for degenerate problems
- One such strategy is Bland's rule: If there are multiple variables with positive coefficients in objective function, always choose the variable with smallest index
- Example: If $z = 2x_1 + 5x_2 - 4x_3$, Bland's rule chooses x_1 as new basic variable since it has smallest index

Simplex Algorithm Phases

- Simplex algorithm has two phases:
  1. Phase I: Compute a feasible basic solution, if one exists
  2. Phase II: Optimize value of objective function ✓
- So far, we talked about the second phase, assuming we already have a feasible basic solution
- However, the initial basic solution might not feasible even if the linear program is feasible

Example of Infeasible Initial Basic Solution

- Consider the following linear program:

$$\begin{array}{lcl} z & = & 2x_1 - x_2 \\ x_3 & = & 2 - 2x_1 + x_2 \\ x_4 & = & -4 - x_1 + 5x_2 \end{array}$$

- What is the initial basic solution?
- Clearly, this solution is not feasible
- Goal of Phase I of Simplex is to determine if a feasible basic solution exists, and if so, what it is

Overview of Phase I

- To find an initial basic solution, we construct an auxiliary linear program L_{aux}
- This auxiliary linear program has the property that we can find a feasible basic solution for it after at most one pivot operation
- Furthermore, original LP problem has a feasible solution if and only if the optimal objective value for L_{aux} is zero
- If optimal value of L_{aux} is 0, we can extract basic feasible solution of original problem from optimal solution to L_{aux}

Constructing the Auxiliary Linear Program

- Consider the original LP problem:

Maximize $\sum_{j=1}^{n} c_j x_j$
Subject to:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad (i \in [1,m])$$
$$x_j \geq 0 \quad (j \in [1,n])$$

- This problem is feasible iff the following LP problem L_{aux} has optimal value 0:

Maximize $-x_0$
Subject to:

$$\sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i \quad (i \in [1,m])$$
$$x_j \geq 0 \quad (j \in [0,n])$$

Justification for Auxiliary LP

Maximize $-x_0$
Subject to:

$$\sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i \quad (i \in [1,m])$$
$$x_j \geq 0 \quad (j \in [0,n])$$

$\Rightarrow$ Suppose x_0 has optimal value 0. Then clearly $a_{ij}x_j \leq b_i$ is satisfied for all inequalities

$\Leftarrow (a)$ Suppose original problem has feasible solution $\vec{x}^*$. Then $\vec{x}^*$ combined with $x_0 = 0$ is feasible solution for L_{aux}.

$\Leftarrow (b)$ Due to the non-negativity constraint, $-x_0$ can be at most 0; thus, this solution is optimal for L_{aux}.

Finding Feasible Basic Solution for L_{aux}

- So far, we argued that original problem L has feasible solution iff L_{aux} has optimal value 0.
- But we still need to figure out how to find feasible basic solution to L_{aux}.
- Next: We'll see how we can find feasible basic solution for L_{aux} after one pivot operation.

Auxiliary Problem in Slack Form

$$z = -x_0$$
$$x_i = b_i + x_0 - \sum_{j=1}^{n} a_{ij}x_j$$

- If all b_i's are positive, basic solution already feasible
- If there is at least some negative b_i, find equality x_i with most negative b_i
- Make x_0 new basic variable, and x_i non-basic
- Claim: After this one pivot operation, all b_i's are non-negative; thus basic solution is feasible

Why is This True?

- Suppose this equality has most negative b_i:

$$x_i = b_i + x_0 - \sum_{j=1}^{n} a_{ij}x_j$$

- Rewrite to make x_0 basic:

$$x_0 = -b_i + x_i + \sum_{j=1}^{n} a_{ij}x_j$$

- Now, $-b_i$ is positive and greater than all other $|b_j|$'s
- Thus, when we plug in equality for x_0 into other equations, their new constants will be positive
- Hence, we find a feasible basic solution after at most one pivot step

Example

- Consider the following linear program from earlier:

$$z = 2x_1 - x_2$$
$$x_3 = 2 - 2x_1 + x_2$$
$$x_4 = -4 - x_1 + 5x_2$$

- Construct L_{aux}:

$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$x_4 = -4 + x_0 - x_1 + 5x_2$$

- Which equation has most negative constant?
- Swap x_4 and x_0:

$$x_0 = 4 + x_4 + x_1 - 5x_2$$

Example, cont

- After pivoting, we obtain the new slack form:

$$\begin{array}{rcl} z & = & -4 - x_4 - x_1 + 5x_2 \\ x_3 & = & 6 - x_1 - 4x_2 + x_4 \\ x_0 & = & 4 + x_4 + x_1 - 5x_2 \end{array}$$

- What is current objective value?
- How can we increase it?
- Which equation constrains x_2 the most?
- Swap x_2 and x_0:

$$x_2 = \frac{4}{5} - \frac{1}{5}x_0 + x_4 + x_1$$

Example, cont

- After pivoting, new slack form:

$$\begin{array}{rcl} z & = & -x_0 \\ x_2 & = & \frac{4}{5} - \frac{x_0}{5} - \frac{x_1}{5} + \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} + \frac{4x_0}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \end{array}$$

- Objective function cannot be increased, so we are done!
- In original problem, objective function was $z = 2x_1 - x_2$
- Since x_2 is now a basic variable, substitute for x_2 with RHS:

$$z = \frac{-4}{5} + \frac{9x_1}{5} - \frac{x_4}{5}$$

- Thus, Phase I returns the following slack form to Phase II:

$$\begin{array}{rcl} z & = & \frac{-4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \\ x_2 & = & \frac{4}{5} - \frac{x_1}{5} + \frac{x_4}{5} \\ x_3 & = & \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \end{array}$$

Summary

- To solve constraints in $T_{\mathbb{Q}}$ (linear inequalities over rationals), we use Simplex algorithm for LP
- Simplex has two phases
- In first phase, we construct slack form such that it has a basic feasible solution
- In second phase, we start with basic feasible solution and rewrite one slack form into equivalent one until objective value can't increase
- Although Simplex is a worst-case exponential, it is more popular than polynomial-time algorithms for LP