CS389L: Automated Logical Reasoning

Lecture 3: Practical SAT solving

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Overview

- Today: How state-of-the-art SAT solvers work
- Many competitive solvers based on DPLL, but extend it in three important ways:
  1. Non-chronological backtracking
  2. Learning from past "mistakes"
  3. Heuristics for choosing variables and assignments

Non-Chronological Backtracking

- Recall basic DPLL: First try assigning p to $\top$; if doesn't work, backtrack to most recent decision level and try $p = \bot$
- Called chronological backtracking but often sub-optimal
- Suppose made assignments $p_1, p_2, \ldots p_{100}$ but discovered p_4 was a bad choice
- Backtracking to decision level associated with p_{100} is stupid...
- In non-chronological backtracking, can go back to earlier decision levels

Learning

- Learning = acquisition of new clauses to prevent similar bad assignments
- For instance, suppose we discover $p_5 = \top, p_{32} = \bot, p_{100} = \top$ is inconsistent, i.e.,

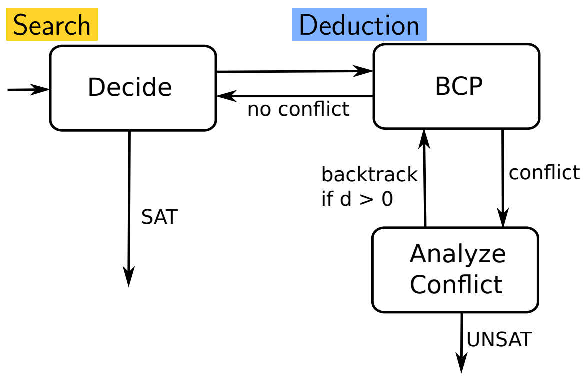
$$\phi \Rightarrow \neg(p_5 \wedge \neg p_{32} \wedge p_{100})$$

- Can add this clause without changing satisfiability (why?)
- Such clauses called conflict clauses $\Rightarrow$ SAT solver has database of conflict clauses

Decision Heuristics

- Basic DPLL chooses variables in random order
- But making assignment to certain variables can make formula much easier to solve!
- Modern solvers use more sophisticated heuristics
- This is something of a black art, but one of the most important elements in SAT solving ...

Architecture of DPLL-Based SAT Solvers



The Plan

- We will talk about BCP and AnalyzeConflict first (related)
- Then: common decision heuristics used in the Decide step
- Finally: Implementation tricks to make all this fast

BCP in SAT Solvers

- Recall: BCP is all possible applications of unit resolution
- SAT solvers remember deductions performed in the BCP process $\Rightarrow$ recorded as implication graph
- First some terminology ...

Some Terminology and Conventions

- Decision variable: variable assigned in the Decide step
- The decision level of a decision variable is the level (order) in which it was assigned
- The decision level of a variable assigned due to BCP is the decision level of the last assigned decision variable
- Important note: Think of assignments as literals: Assignment $p = \top$ is literal p; assignment $p = \bot$ as literal $\neg p$
- Also: An assignment corresponds to a new unit clause added to our set of clauses

Decision Level Example

$$(\neg x_1 \vee x_2) \wedge (\neg x_3 \vee \neg x_4)$$

- Decide assigns $x_1 = \top \Rightarrow x_1$ decision var at level 1
- BCP yields:
- Decision level of x_2?
- Decide next assigns $x_4 = \top$. BCP deduces:
- x_4 decision variable with decision level:
- x_3's decision level:

Implication Graph

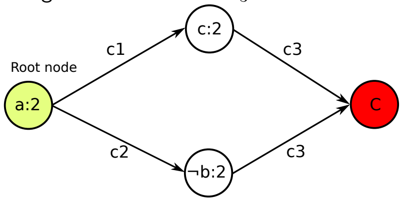
- An implication graph is a labeled directed acyclic graph
- Nodes: literals in the current partial assignment
- Node labels: Indicate assignment and decision level.
- Example: Node labeled $\neg x : 3$ means variable x was assigned to $\bot$ at decision level 3
- Edges from $l_1, \ldots l_k$ to l labeled with c: Assignments $l_1, \ldots, l_k$ caused assignment l due to clause c during BCP
- A special node C is called the conflict node.
- Edge to conflict node labeled with c: current partial assignment contradicts clause c.

Implication Graph Example

- Consider the following set of clauses:

$$c_1 : (\neg a \vee c) \quad c_2 : (\neg a \vee \neg b) \quad c_3 : (\neg c \vee b)$$

- Assume *Decide* assigned $a = \top$ at decision level 2
- BCP yields:
- Assignment contradicts c_3!



Another Example

- Consider the following clauses:

  $c_1 : (\neg a \lor c) \quad c_2 : (\neg c \lor \neg a \lor b) \quad c_3 : (\neg c \lor d) \quad c_4 : (\neg d \lor \neg b)$

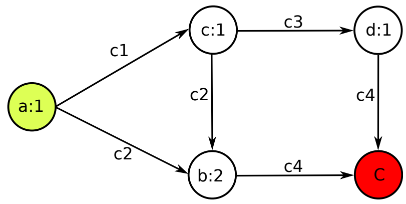
- Suppose *Decide* assigned $a = \top$ at decision level 1
- Using clause c_1, BCP yields:
- Using clause c_2, BCP yields:
- Using clause c_3, BCP yields:
- Assignment $b = \top, d = \top$ contradicts:

Example cont.

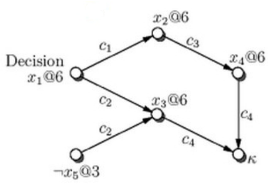
- Consider the following clauses:

  $c_1 : (\neg a \lor c) \quad c_2 : (\neg c \lor \neg a \lor b) \quad c_3 : (\neg c \lor d) \quad c_4 : (\neg d \lor \neg b)$

- Suppose *Decide* assigned $a = \top$ at decision level 1
- Resulting implication graph:



Example 3



- Based on this implication graph, what is c_4?
- What is c_3?
- What is c_1?
- What is c_2?

Implication Graph Properties

- Root nodes in the implication graph correspond to what kind of variables?
- Edges and internal nodes arise due to BCP
- If literal l has incoming edge labeled c, what do we know about c?
- If literal l has outgoing edge labeled c, what do we know about c?

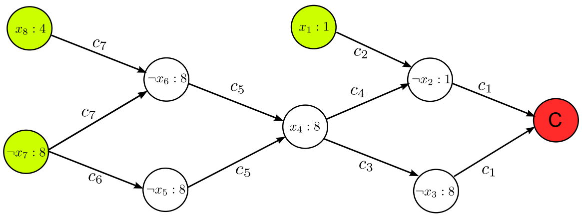
Analyzing Conflicts

- Point of implication graph: analyze conflict
- AnalyzeConflict has two goals:
  1. Learn new conflict clauses
  2. Figure out what level to backtrack to

Conflict Clauses

- A conflict clause is a clause implied by the original formula
- Point of conflict clause: Prevent bad partial assignments by deriving contradiction as quickly as possible
- Question: To achieve this goal, are small or large conflict clauses better?
- Answer: Small ones because the smaller the clause, the quicker BCP forces variable assignments, and the quicker we derive contradictions!
- The implication graph is very useful for deriving small clauses implied by the original formula!

Using Implication Graph to Analyze Conflicts



- What can we say about source of conflict based on this (partial) implication graph?
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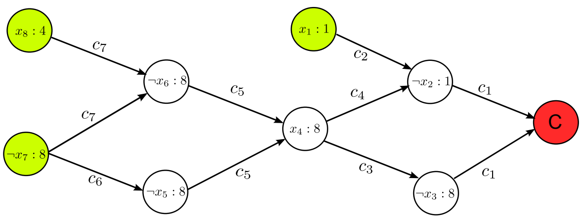
- Are other decision variables relevant to conflict?

Simple Strategies to Derive Conflict Clause

- One way to derive conflict clause: The negation of current partial assignment
- Another way: Conjoin all literals associated with root nodes reaching conflict node, use negation as conflict clause
- Question: Which one is better?

Using Implication Graph to Analyze Conflicts



- In this example, this would yield:
- c' prevents the same partial assignment in the next step

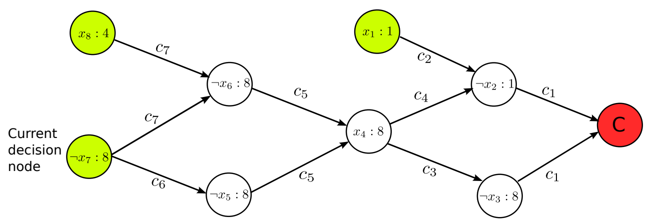
Analyzing Conflicts

- This strategy is one of the earliest strategies proposed for inferring conflict clauses (e.g., the GRASP SAT solver)
- But people have improved upon this; possible to derive even better conflict clauses!
- A key concept is unique implication points

Unique Implication Point

- A node N in the implication graph is a unique implication point (UIP) if all paths from current decision node to the conflict node must go through N
- Is the current decision node a UIP?
- Can there be multiple unique implication points?
- First unique implication point: UIP closest to conflict node

UIP Example

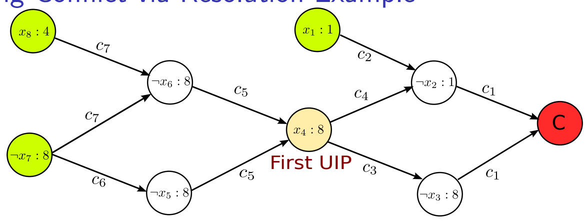


- Which nodes are UIP's?
- Which node is first UIP?

Using UIP and Resolution for Deriving Conflict Clause

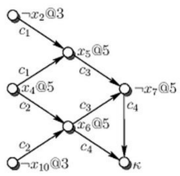
- Common heuristic to infer conflict clauses: Start with clause labeling incoming edge to conflict node, derive new clauses via resolution until we find literal in first UIP
- Specifically: In current clause c, find last assigned literal l in c.
- Pick any incoming edge to l labeled with clause c'.
- Resolve c and c'.
- Set current clause be resolvent of c and c'.
- Repeat until current clause contains negation of the first UIP literal (as the single literal at current decision level)

Analyzing Conflict via Resolution Example



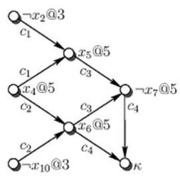
- What is c_1?
- Last assigned literal in c_1:
- Clause c_3 labeling incoming edge:
- Resolve c_1 and c_3:
- $\neg x_4$ only literal from decision level 8 $\Rightarrow$ $x_2 \vee \neg x_4$ conflict clause

Another Example



- What is the first UIP?
- Start with clause c_4:
- Suppose we pick $\neg x_7$
- Clause on incoming edge to $\neg x_7$:
- Resolve c_3, c_4:
- Suppose x_6 assigned later, pick x_6
- Clause on incoming edge:
- Resolve current clause with c_2:

Another Example, cont.



- Current clause:
- Are we done?
- Pick last assigned literal: x_5
- Incoming edge to x_5:
- Resolve with current clause:
- Are we done?
- New conflict clause: $x_2 \vee \neg x_4 \vee x_{10}$

Why is this correct?

- Why are the clauses obtained this way implied by formula?
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- Unclear if there is a deep reason why this works well, but seems effective in practice . . .

Backtracking

- Recall: AnalyzeConflict has two goals.
- First goal: Deriving conflict clauses ✓
- Second goal: Figure out what level to backtrack to
- Backtrack to level d means delete all variable assignments made after level d (but assignments at level d not deleted)

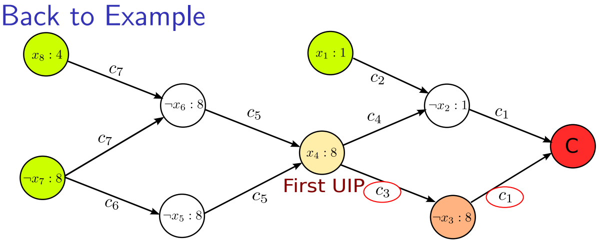
Backtracking and Asserting Clauses

- A good strategy: We want to backtrack to a level that makes conflict clause c an asserting clause in the next step
- Asserting clause is a clause with exactly one unassigned literal
- Hence, if we make c an asserting clause, BCP will force at least one assignment

Choosing Backtracking Level

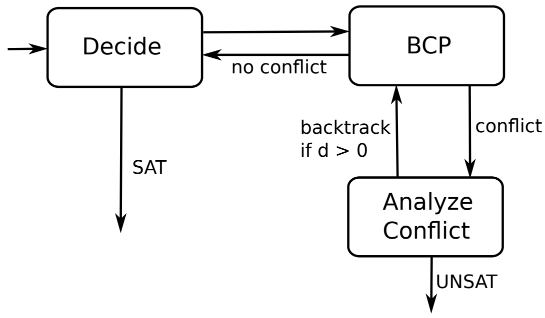
- Question: If we want to make conflict clause c an asserting clause in the next step, what level should we backtrack to?
- Answer:
- Since conflict clause contains only one literal, say l', from the first highest decision level, backtracking to d will assert l'!

Going Back to Example



- Recall: We obtained the conflict clause $x_2 \vee \neg x_4$
- What level do we backtrack to?
- What do we delete in the graph?
- After we add $x_2 \vee \neg x_4$ to clause database, BCP implies:
- Different assignment than before!

Recall: SAT Solver Architecture



- Decision heuristics for choosing variable order and truth assignment

Decision Heuristics

- Important part of SAT solvers, but something of a black art
- Can come up with hundreds of heuristics with varying tradeoffs
- We'll only talk about two:
  1. dynamic largest individual sum (DLIS)
  2. variable state independent decaying sum (VSIDS)

Dynamic Largest Individual Sum (DLIS)

- This heuristic chooses the literal that satisfies the largest number of currently unsatisfied clauses.
- Example: $(x_1 \vee \neg x_2) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3)$
- What assignment would DLIS pick for this formula? (assuming no assignments so far)
- How is this heuristic is dynamic?
- Thus, overhead can be high and must be implemented carefully to minimize bookkeeping

Variable State Independent Decaying Sum (VSIDS)

- Similar to DLIS, but tries to reduce overhead and favor literals involved in conflicts (i.e. conflict-driven)
- Count number of clauses in which the literal appears, but disregard if the clause it appears in is satisfied or not
- Specifically, initialize the score of each literal to the number of clauses in which literal appears
- Every time we add a conflict clause involving literal l, increase the score of that literal by 1
- Periodically divide scores of all literals by 2
  $\Rightarrow$ decaying sum

Variable State Independent Decaying Sum (VSIDS), cont.

- Favors literals involved in conflicts
- If a literal doesn't appear in recent conflict, its score will decay over time
- On the other hand, if literal appears in recent conflict, its score will be increased, so its score won't decay as much
- Much cheaper compared to DLIS because we don't need to scan all clauses to figure out which ones are satisfied
- Introduced in the CHAFF SAT solver from Princeton, written by undergrads!

Implementation Tricks

- To build competitive SAT solvers, it is important to minimize overhead of implementing Decide, BCP, and Analyze Conflict
- Very important because SAT solver might be searching through hundreds of thousands of assignments!
- We'll talk about two issues:
  1. number of conflict clauses
  2. trick to perform BCP fast: watch literals

Conflict Clauses

- Recall: After analyzing conflict, we add new conflict clause to our clause database
- Pro: Conflict clauses quickly block bad assignments and prevent future mistakes
- Con: More clauses = more overhead

$\Rightarrow$ Tradeoff between conflict prevention and minimizing overhead

Conflict Clauses, cont.

- For this reason, many SAT solvers do not keep all the conflict clauses they derive
- For example, they put a limit on the number of conflict clauses they derive
- Typically, keep most recent conflict clauses since they are most relevant to current part of search space

Implementing BCP

- Implementing BCP efficiently is very important because SAT solvers spend a lot of time doing BCP
- Naive implementation of BCP: Requires scanning all currently unsatisfied clauses
- But industrial SAT contain hundreds of thousands of clauses, so scanning all unsatisfied clauses too expensive!
- A more intelligent implementation: Keep mapping from each literal to all clauses in which each literal appears (because we perform unit resolution after each variable assignment)
- But this is still very expensive because typically each literals appears in many clauses

The Trick: Watch Literals

- Modern SAT solvers use a much more clever trick to perform BCP fast: watch literals
- Observe: Ultimate purpose of BCP is to figure out which variable assignments imply which others
- Question: If we are performing unit resolution between l and clause $c = (\neg l \vee l_1, \ldots \vee l_k)$, under what condition will a new assignment be implied?
- Answer:
- Idea: Suffices to look at clauses that have at most two unassigned literals!

Watch Literals

- Select two unassigned literals in each unsatisfied clause as watch literals
- If a watch literal is assigned and clause has other unassigned literals, choose any unassigned literal in clause to be new watch literal
- If a watch literal is assigned and there are no other unassigned non-watch literals left, BCP implies an assignment to the only remaining watch literal!

Watch Literals, cont.

- **Upshot:** To determine if assignment l implies new assignment, only look at those clauses in which $\neg l$ appears as a watch literal
  - If $\neg l$ does not appear, we can't perform unit resolution
  - If $\neg l$ appears but is not a watch literal, then clause has more than two unassigned literals $\Rightarrow$ won't imply new assignment!
- Yielded huge improvement in SAT solver performance!

Practical SAT Solving Summary

- Modern SAT solvers extend DPLL in three ways: non-chronological backtracking, conflict clause learning, decision heuristics, engineering tricks (watch literals)
- Referred to as CDCL: conflict-driven clause learning
- Many competitive SAT solvers based on CDCL