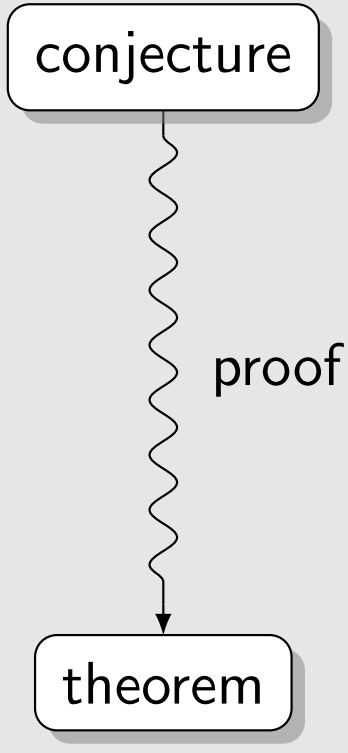
# Fast LCF-Style Proof Reconstruction for Z3

Sascha Böhme and Tjark Weber



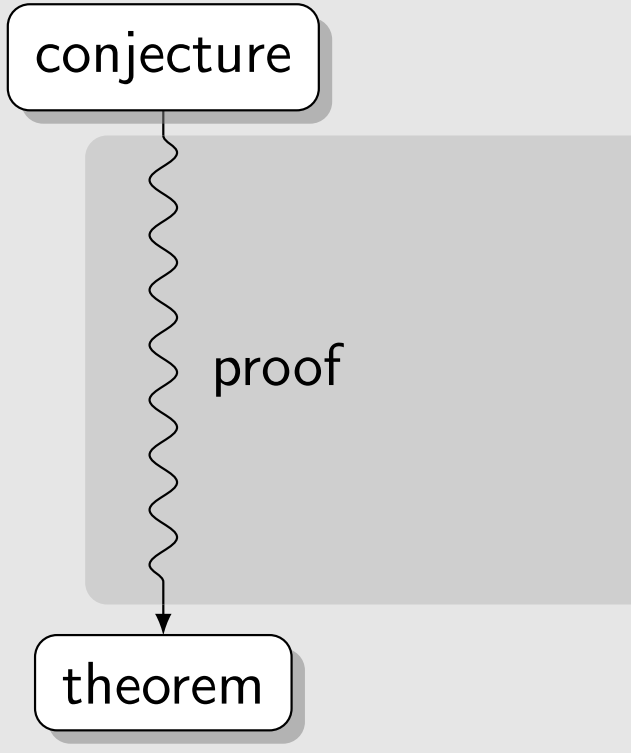
July 14, 2010

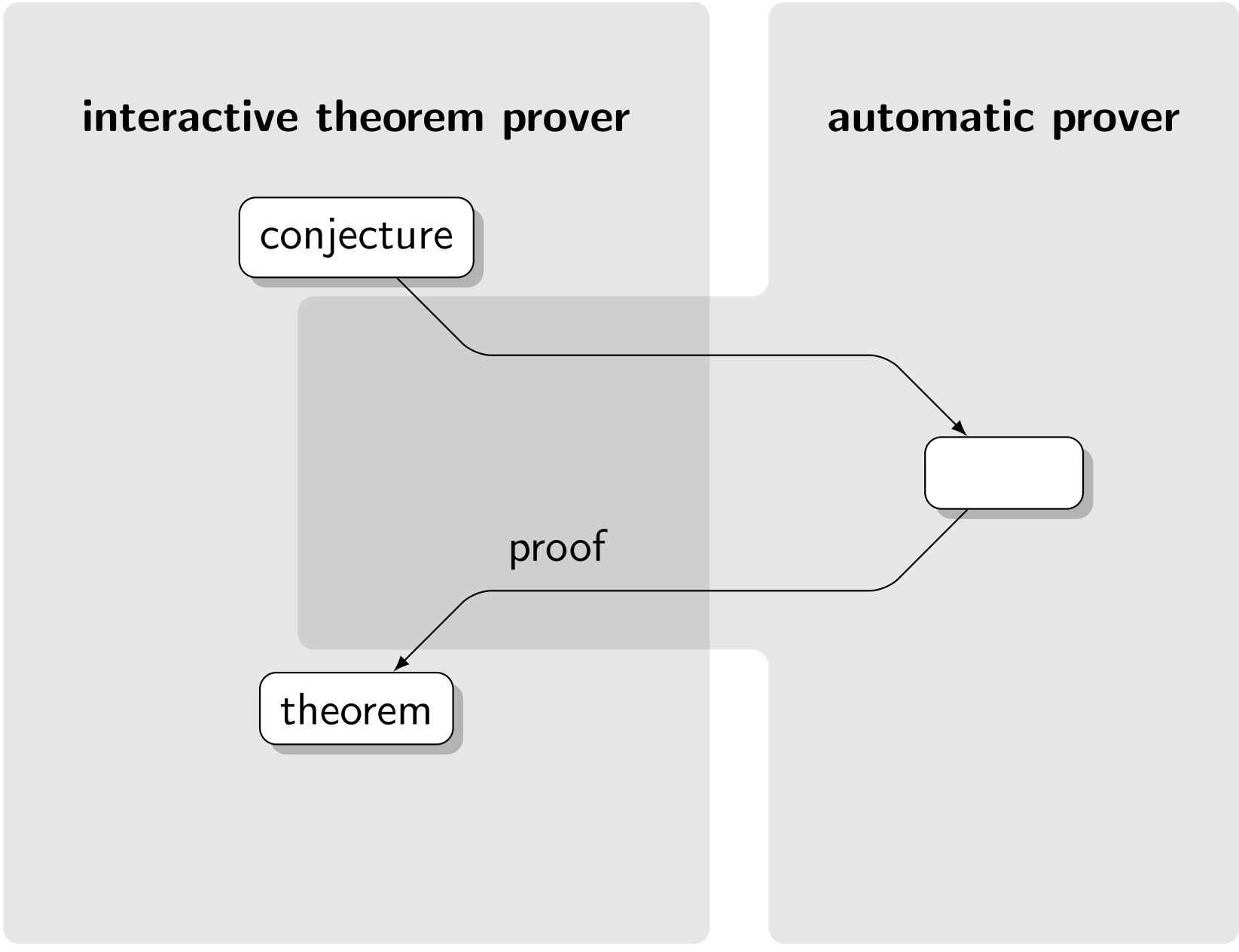
## interactive theorem prover



**interactive theorem prover**

**automatic prover**

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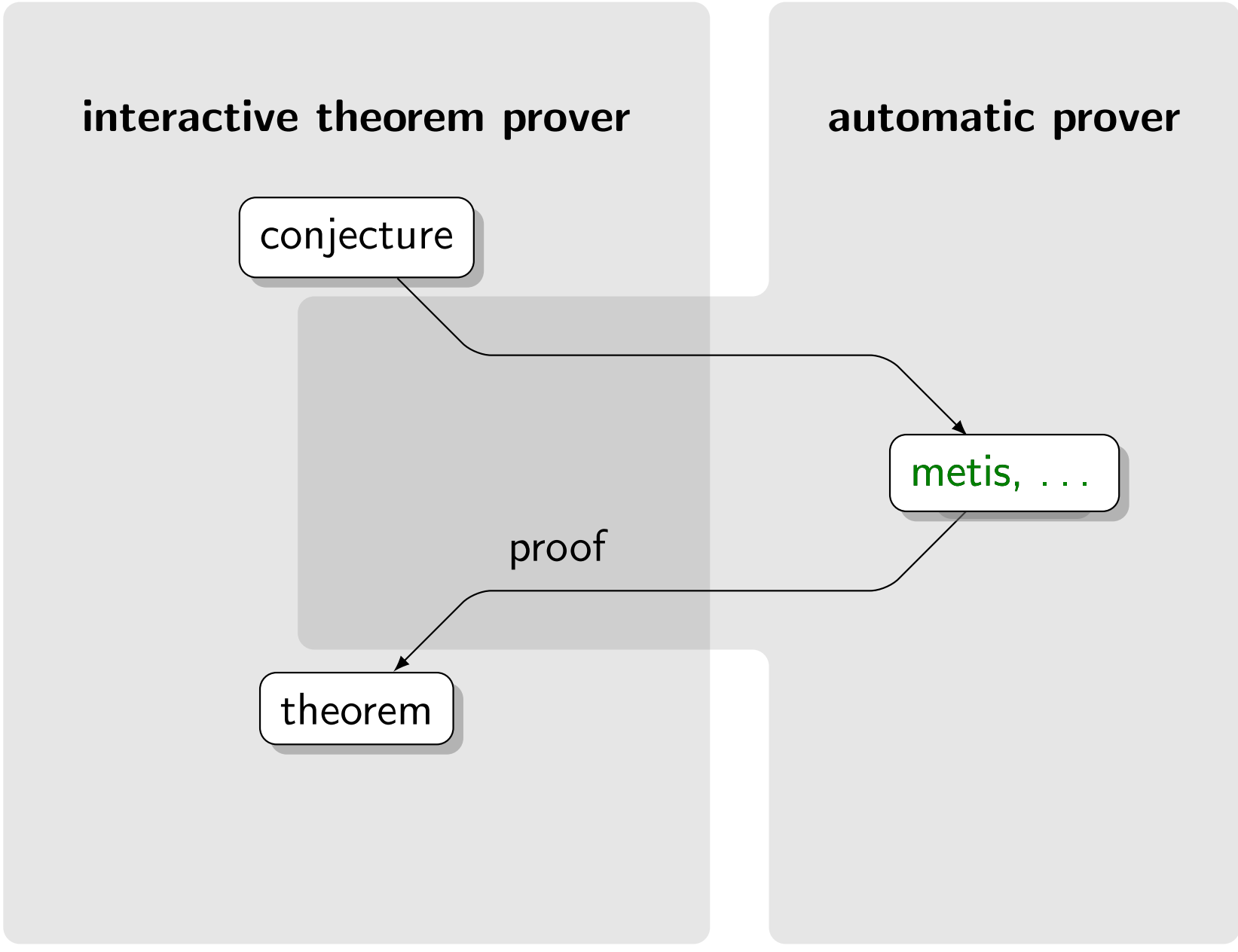
automatic prover
conjecture
proof
theorem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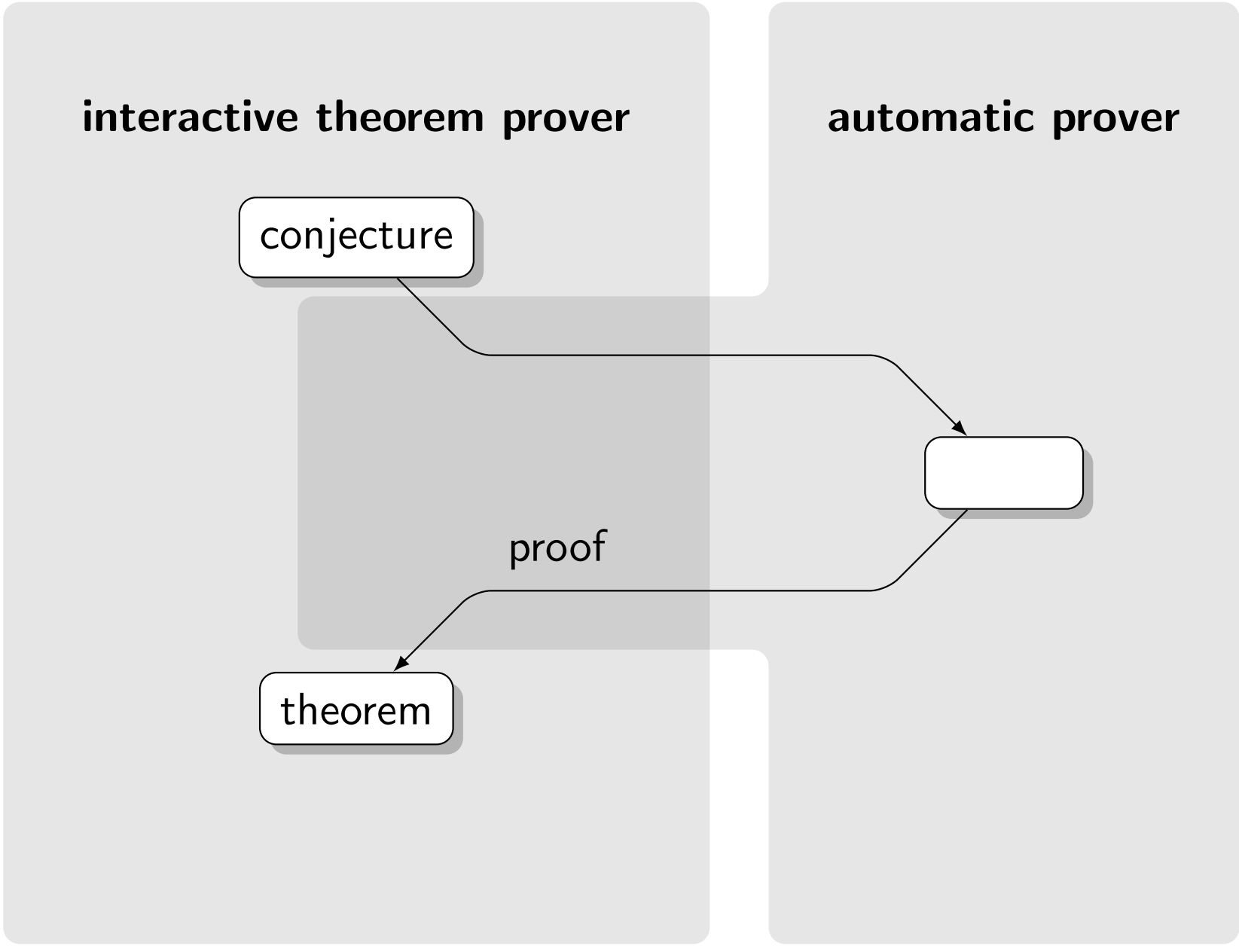
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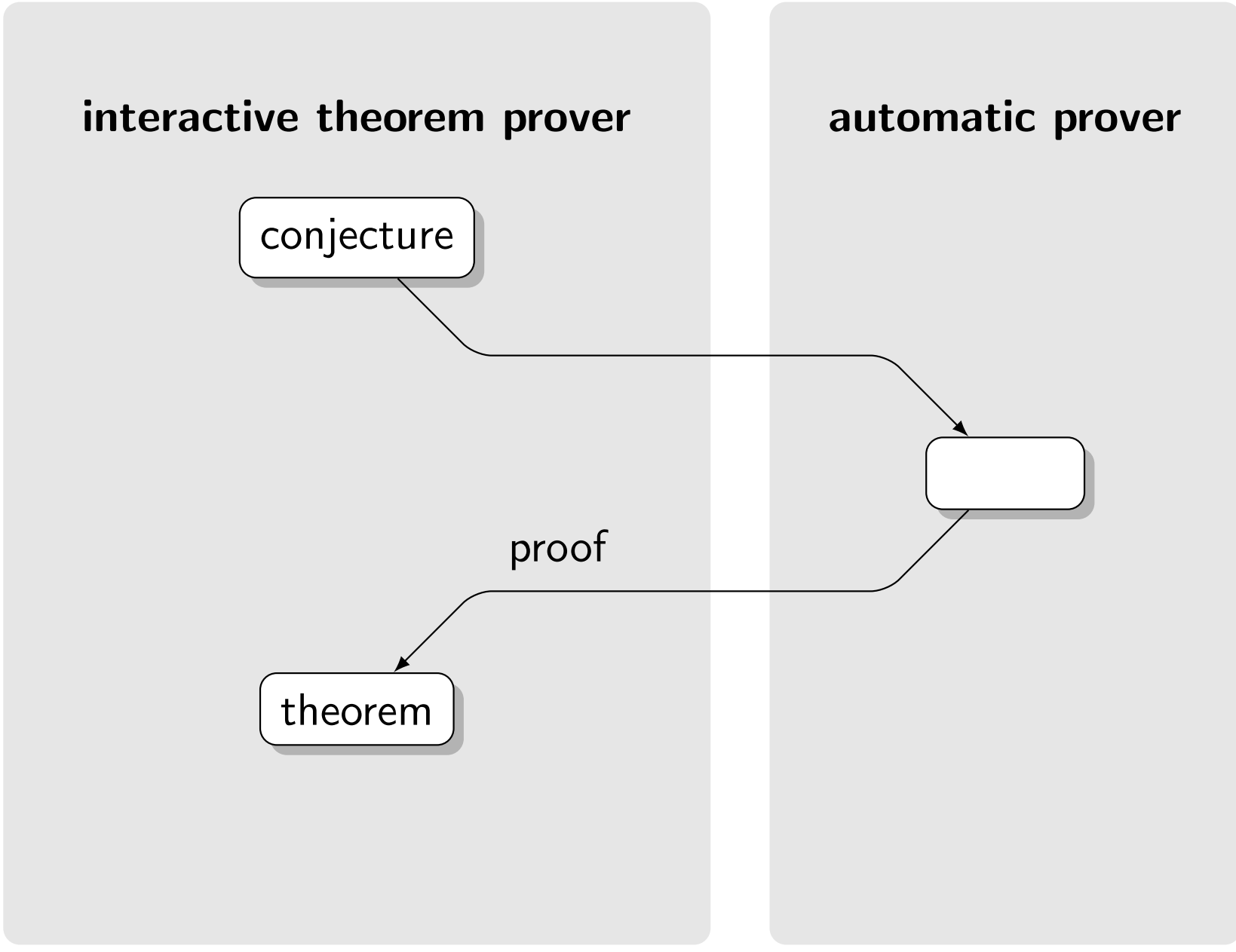
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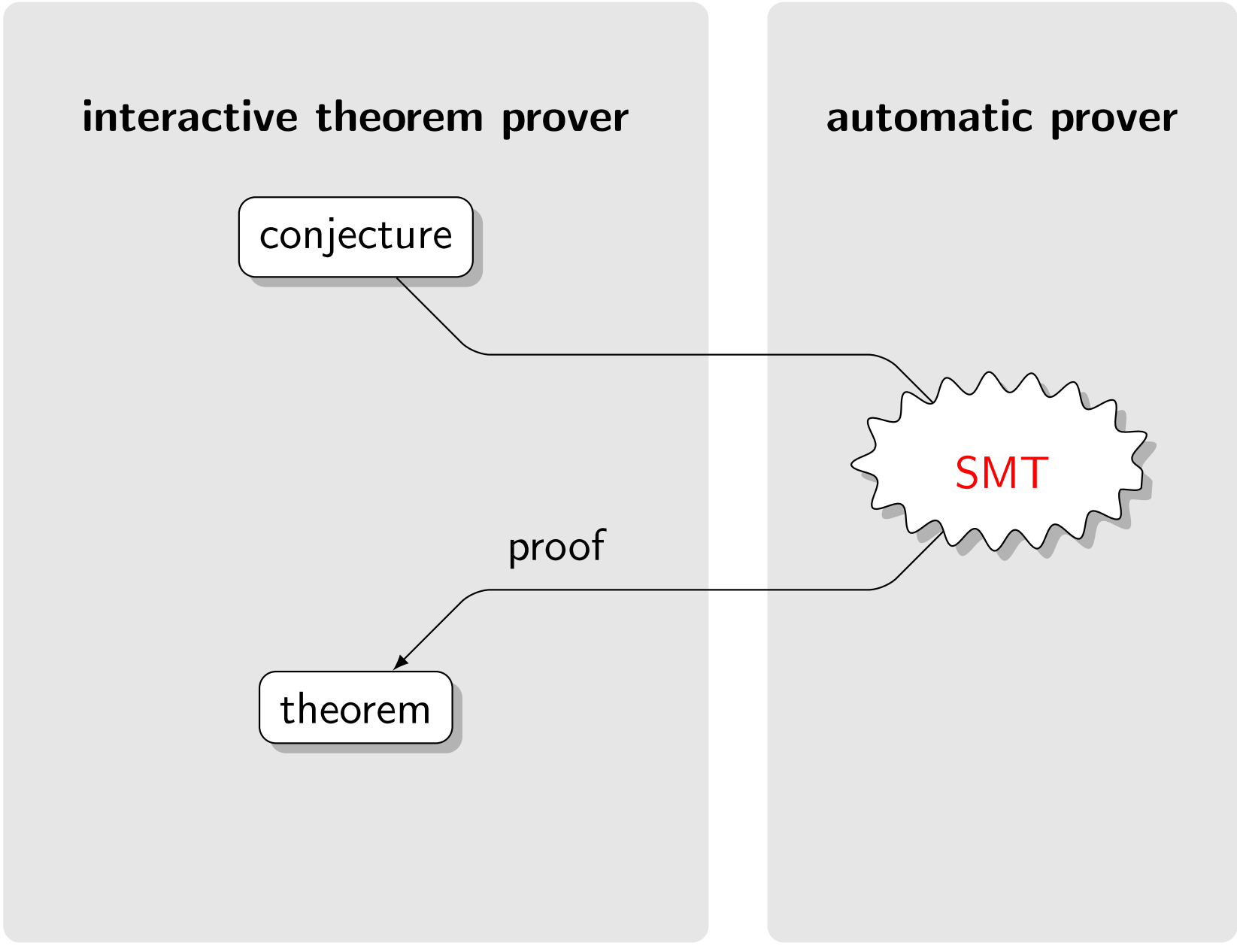
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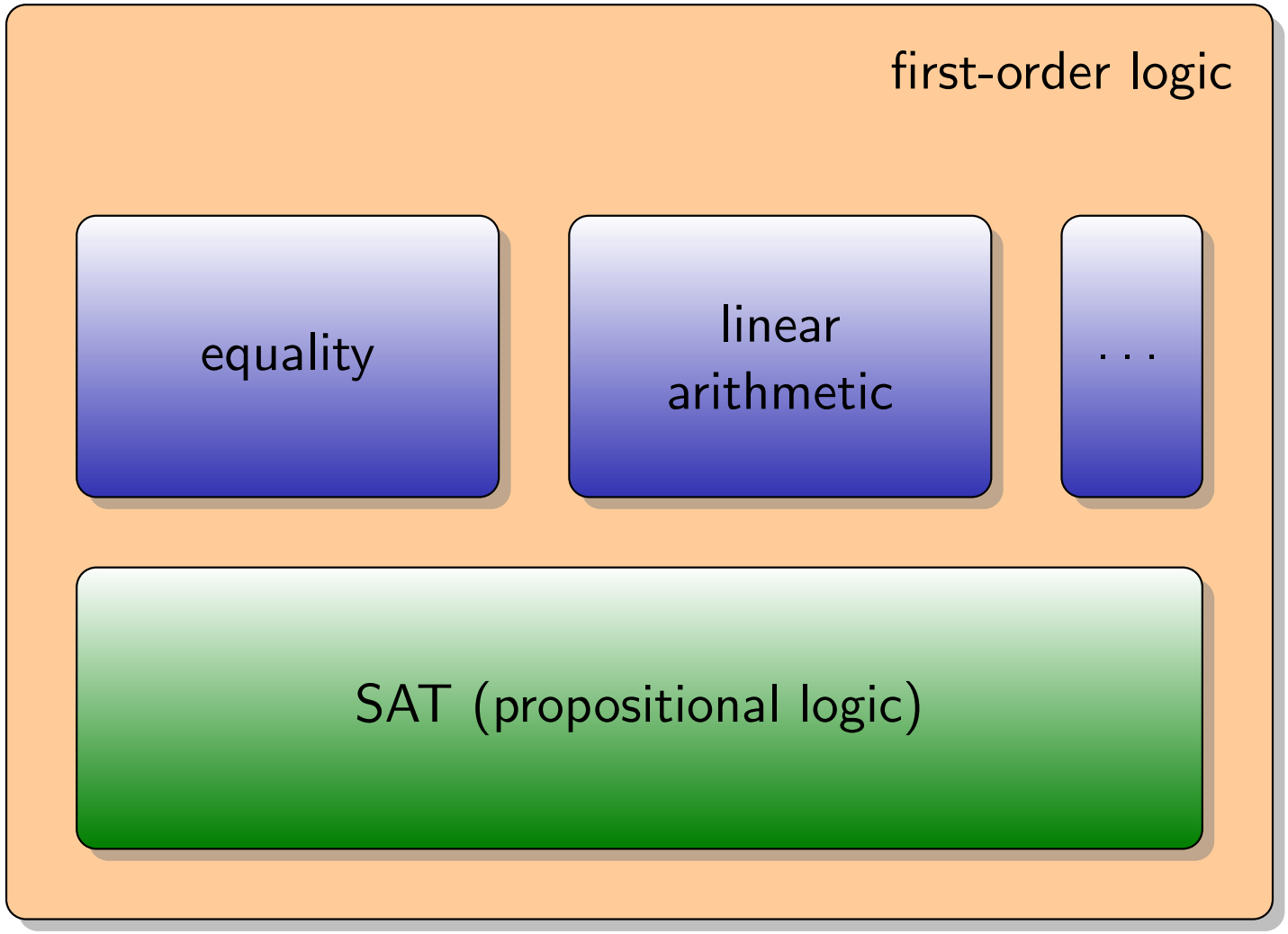
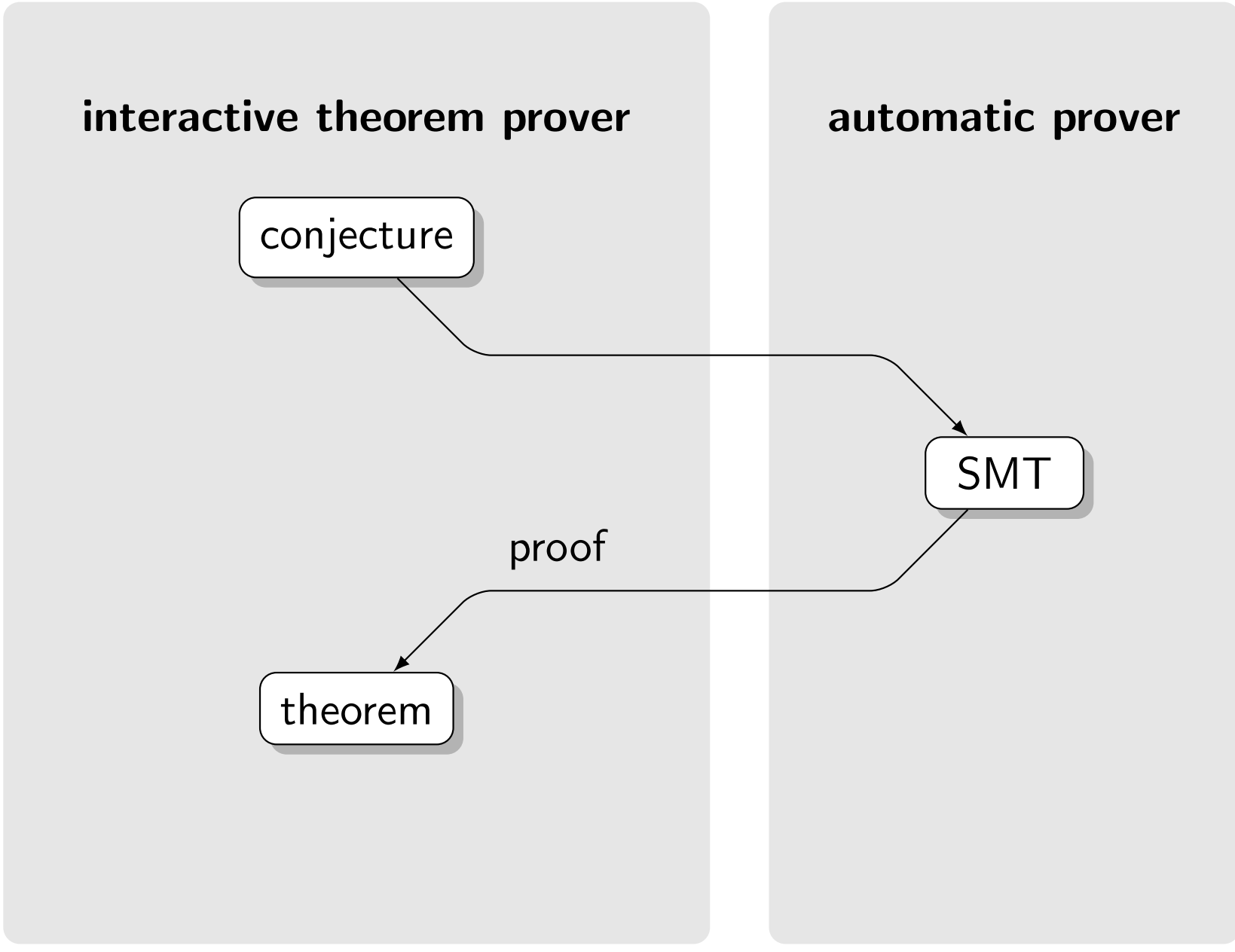
# SMT: Satisfiability Modulo Theories

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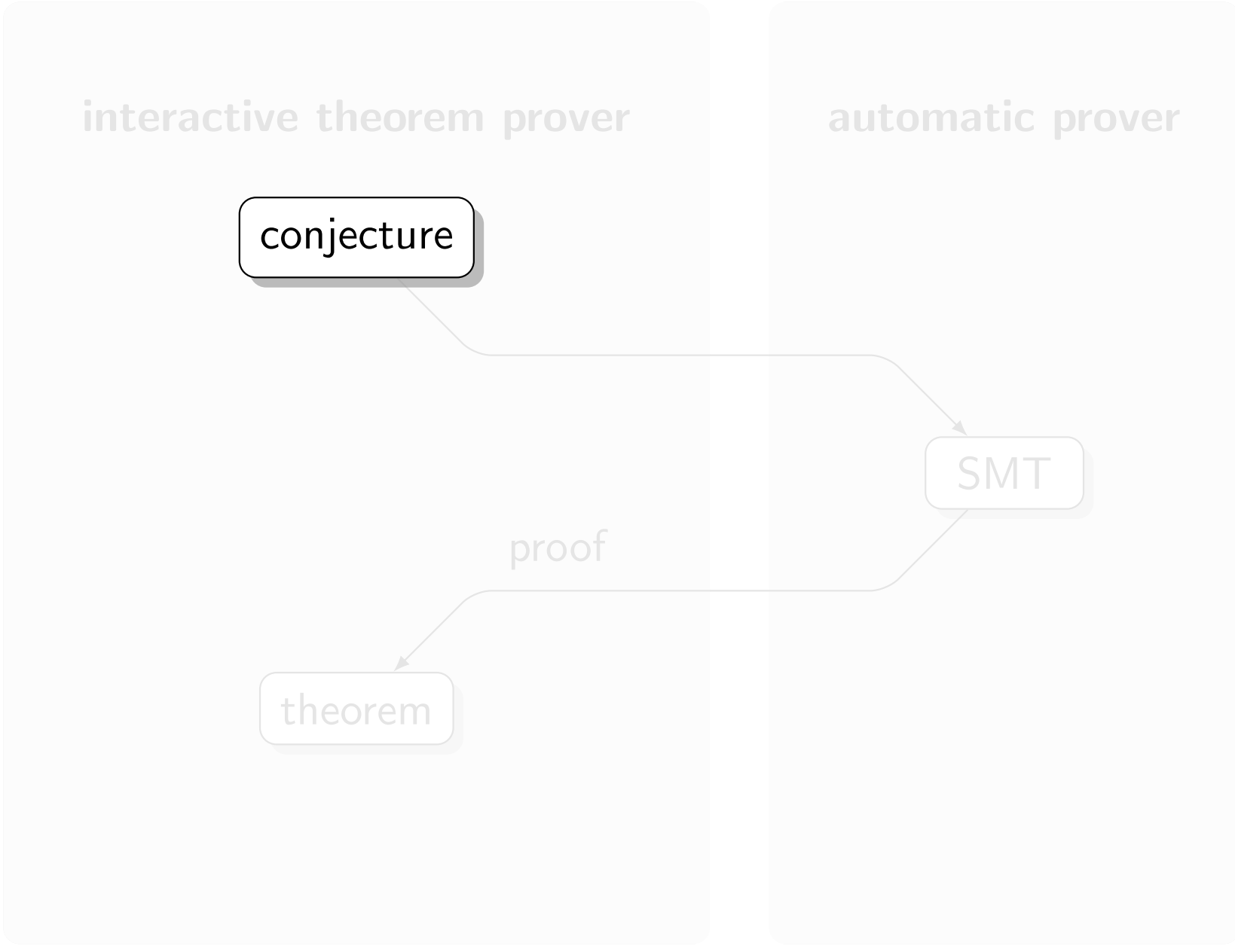
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interactive theorem prover

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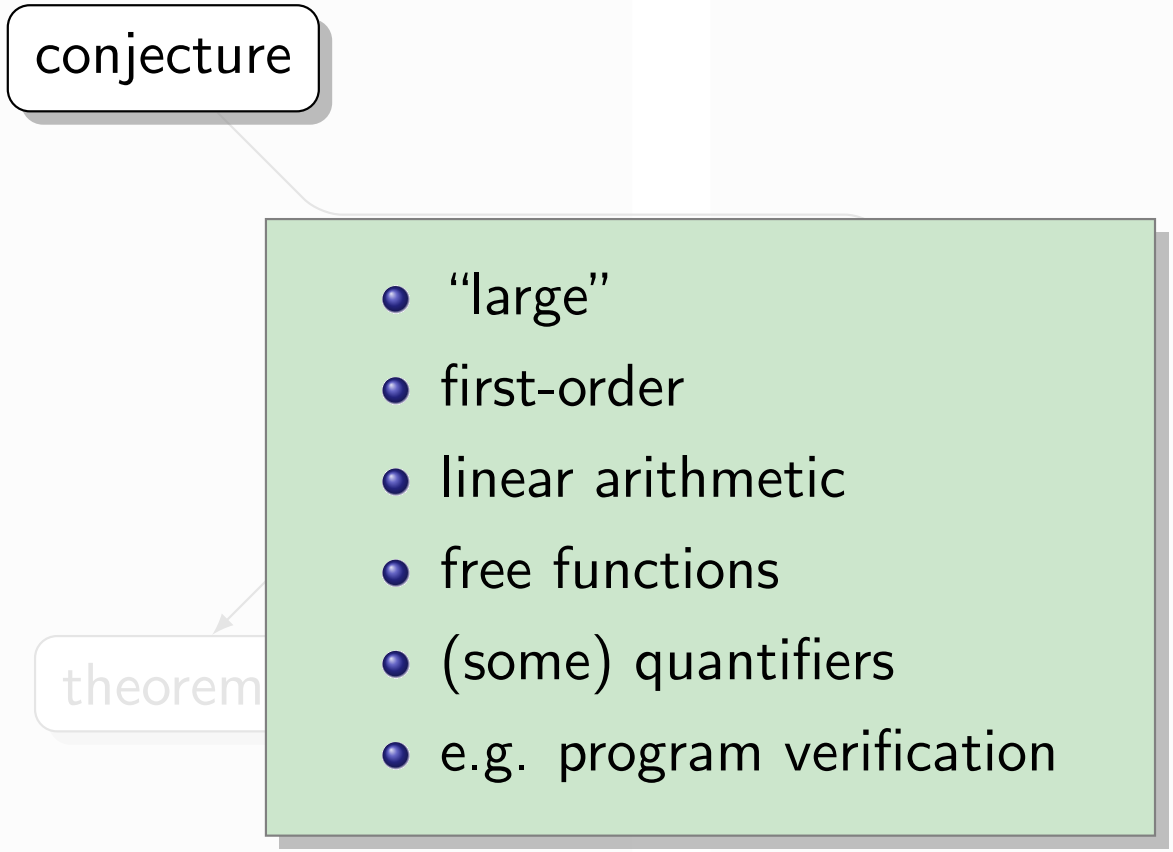

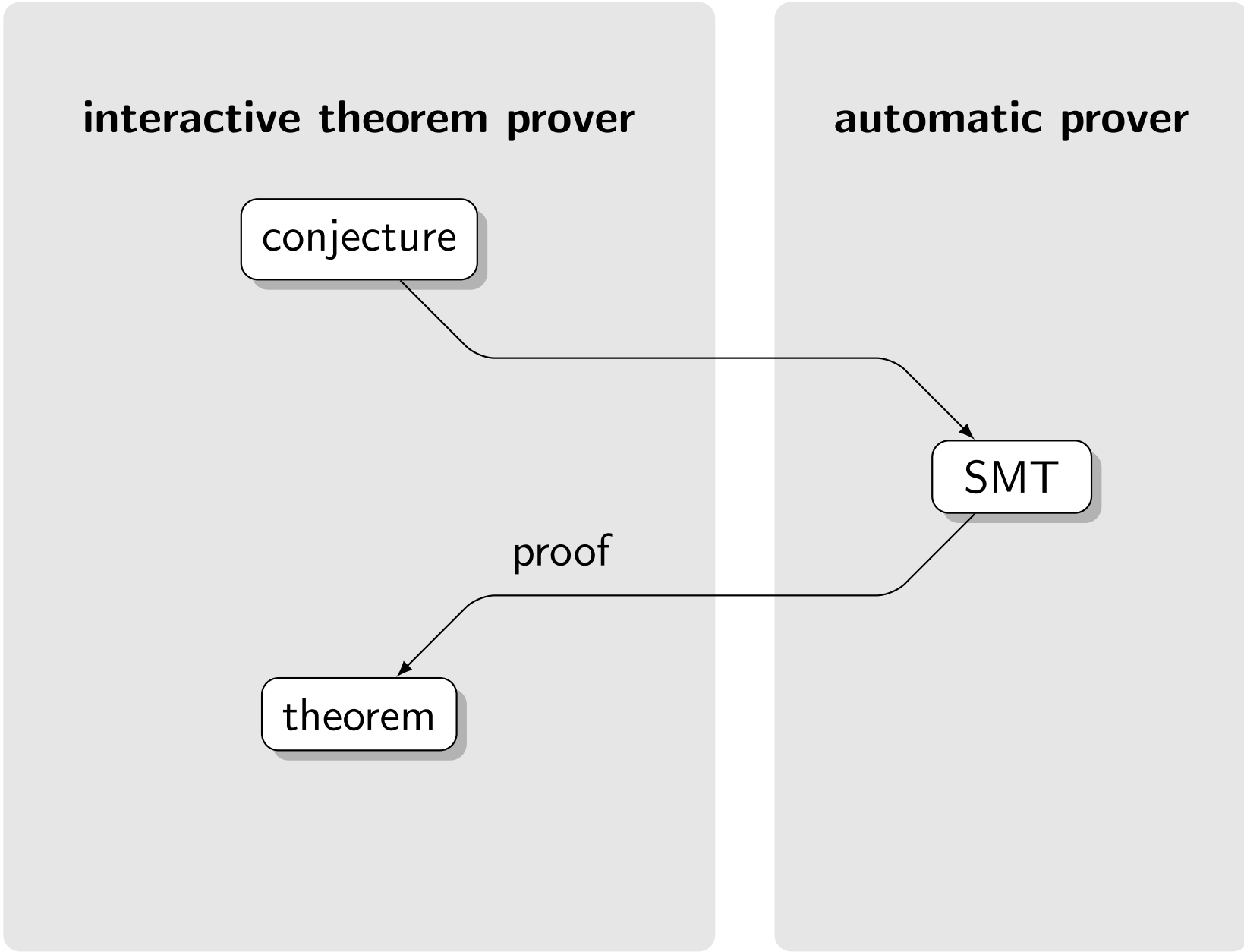
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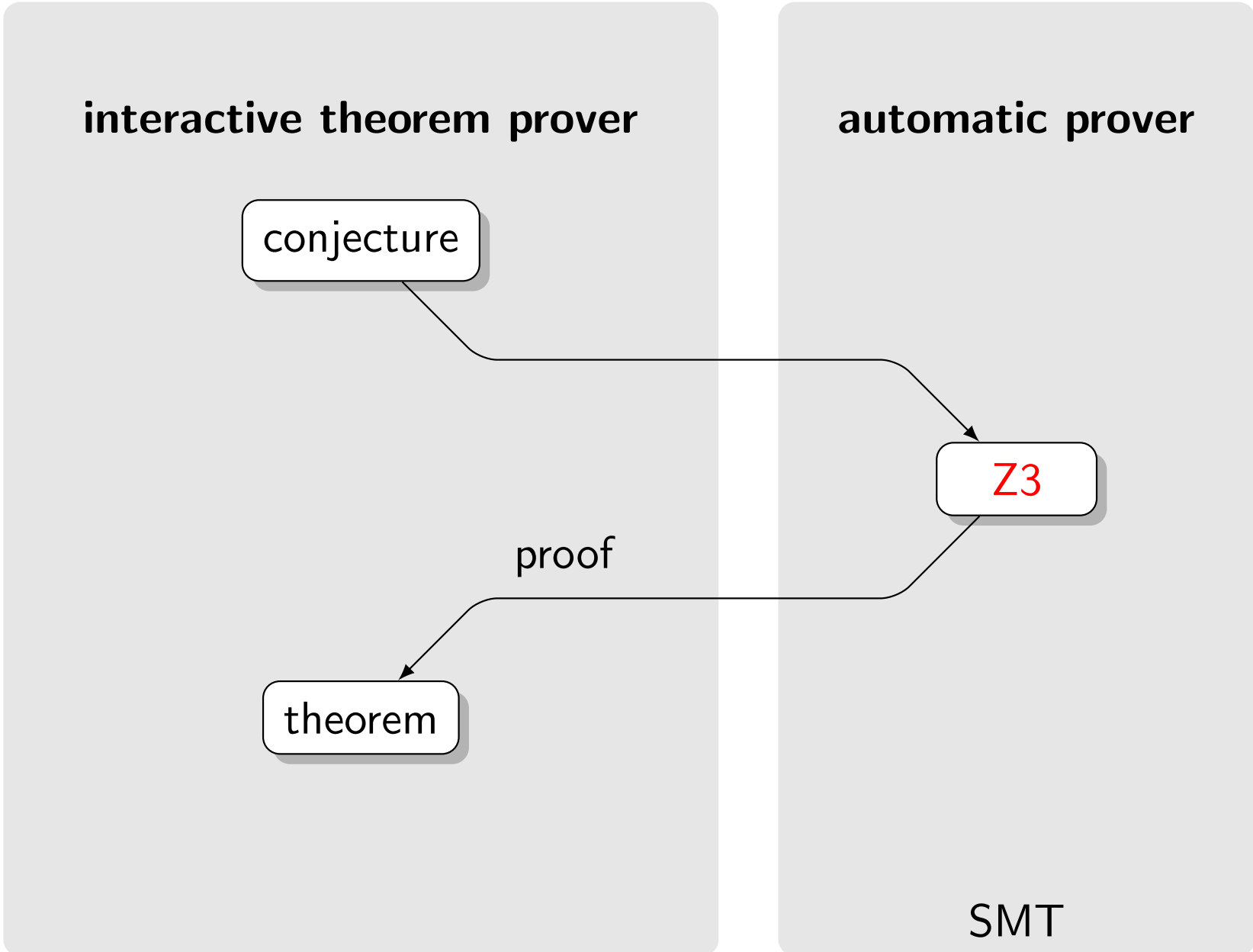
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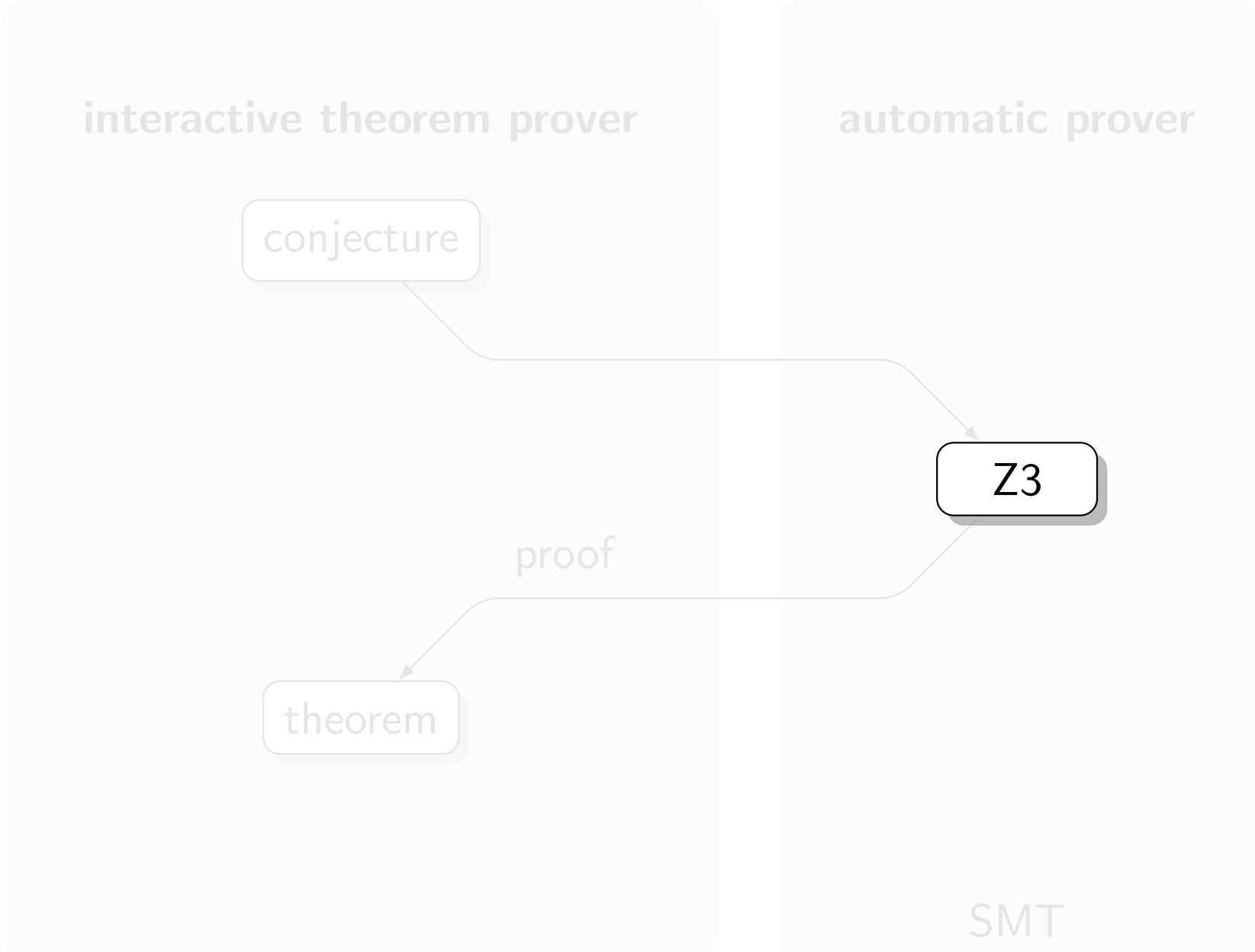
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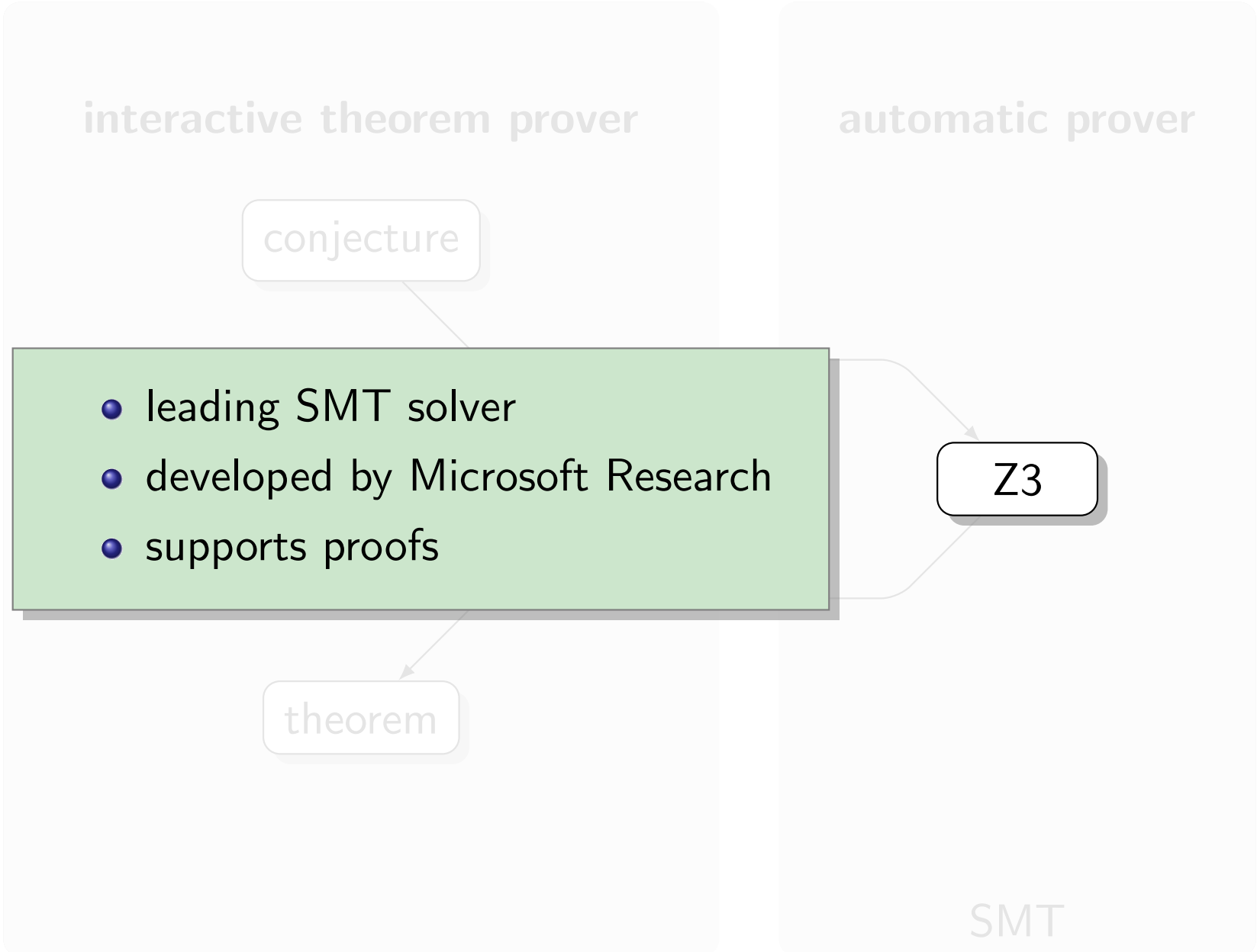
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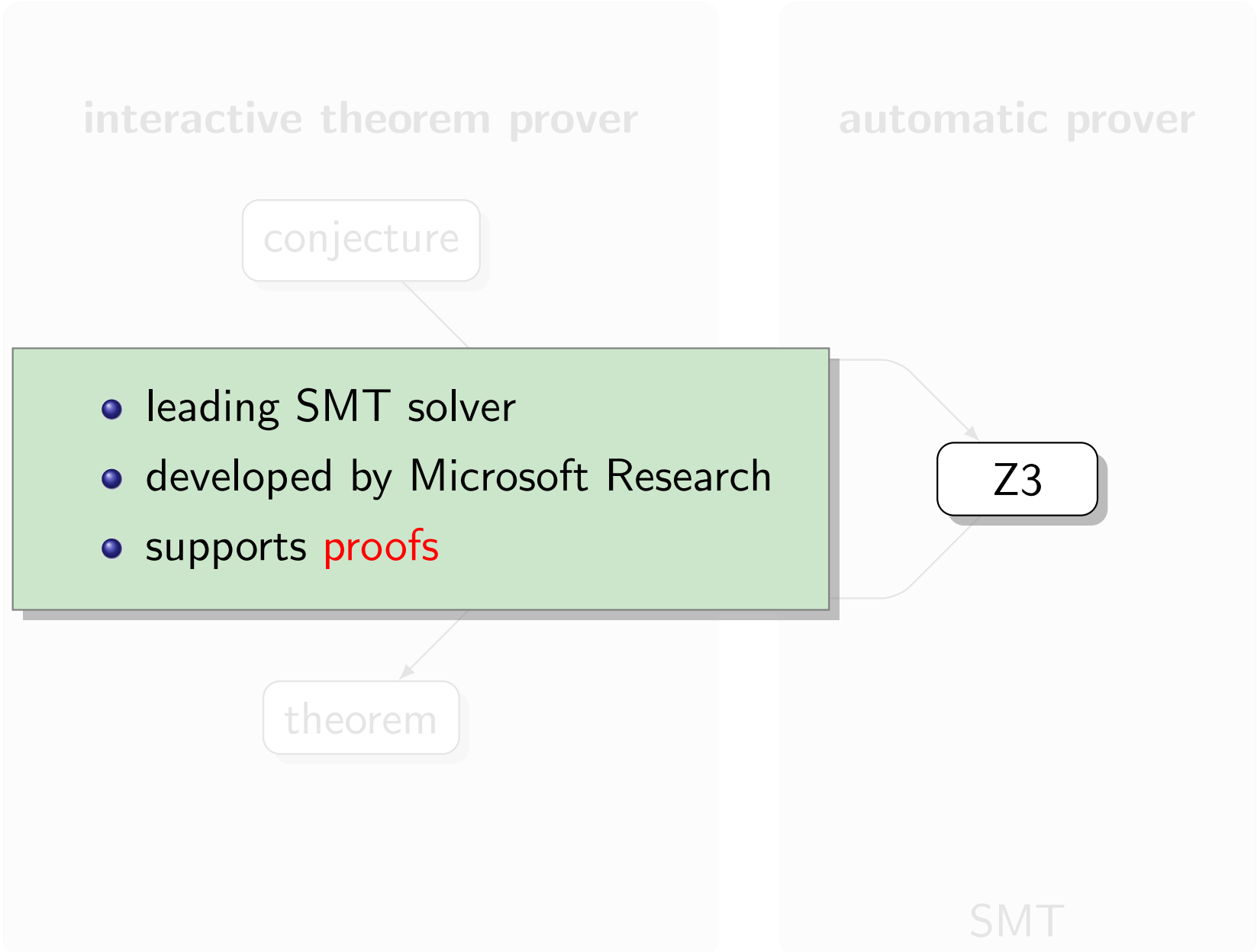
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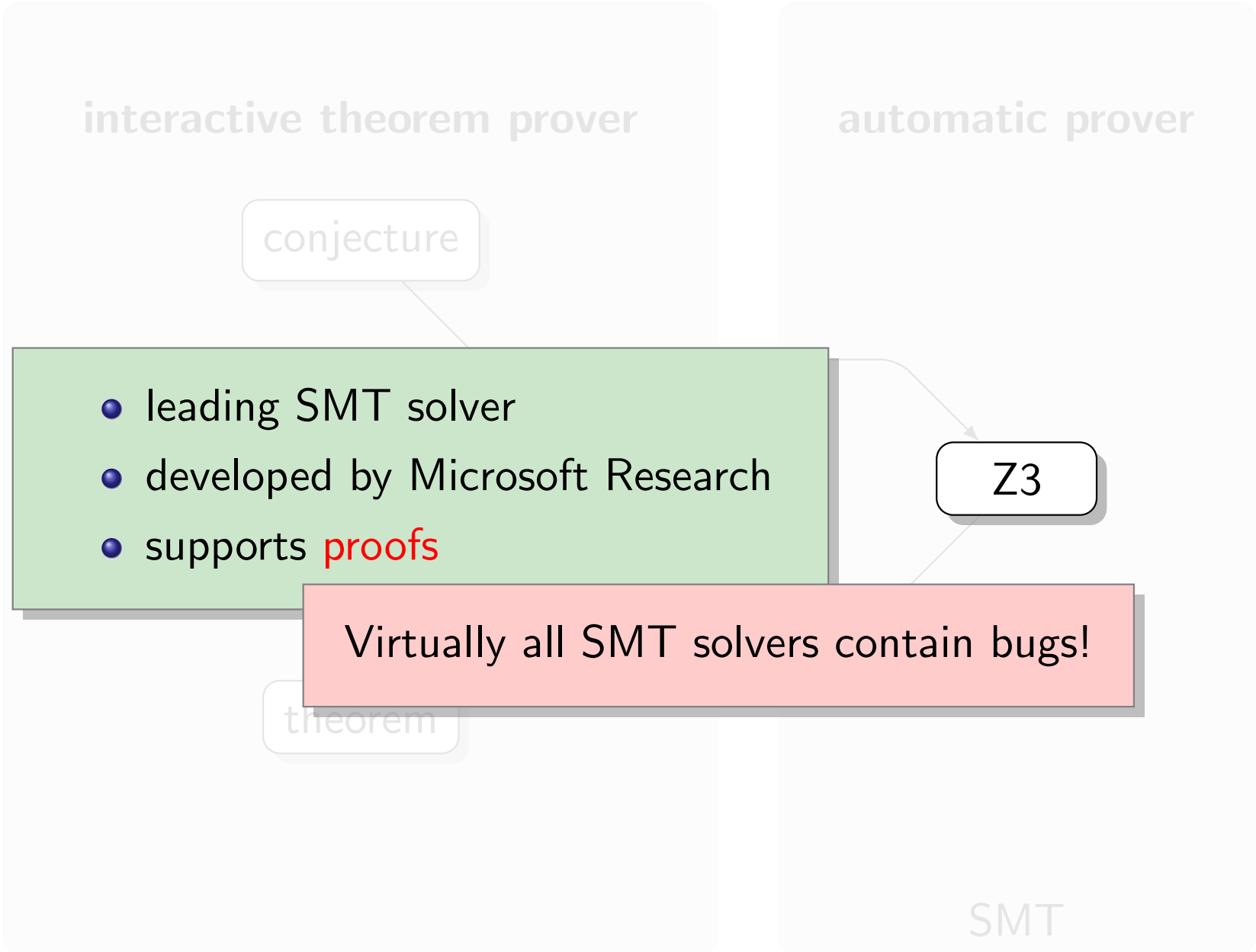
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Virtually all SMT solvers contain bugs!
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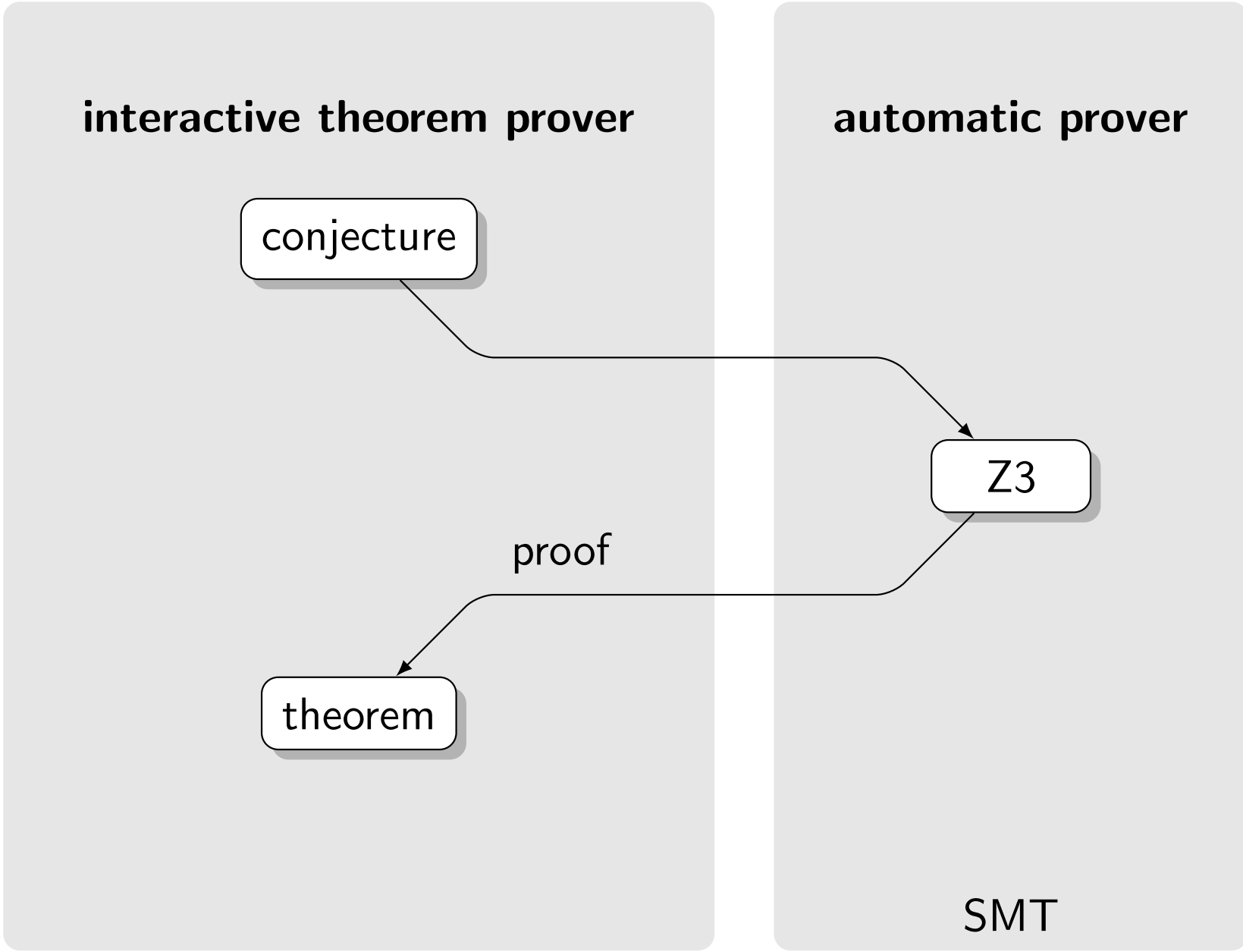
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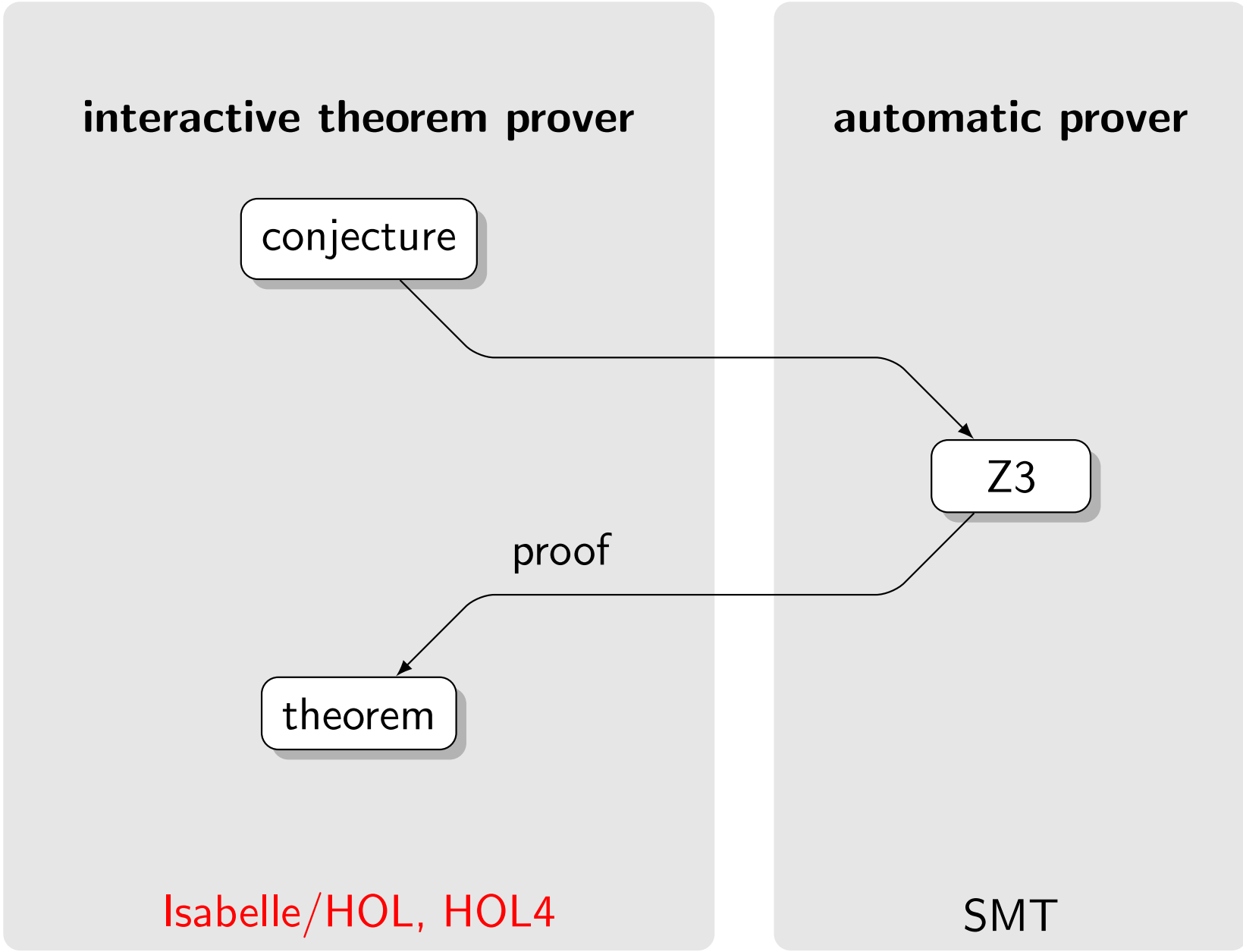
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Isabelle/HOL, HOL4
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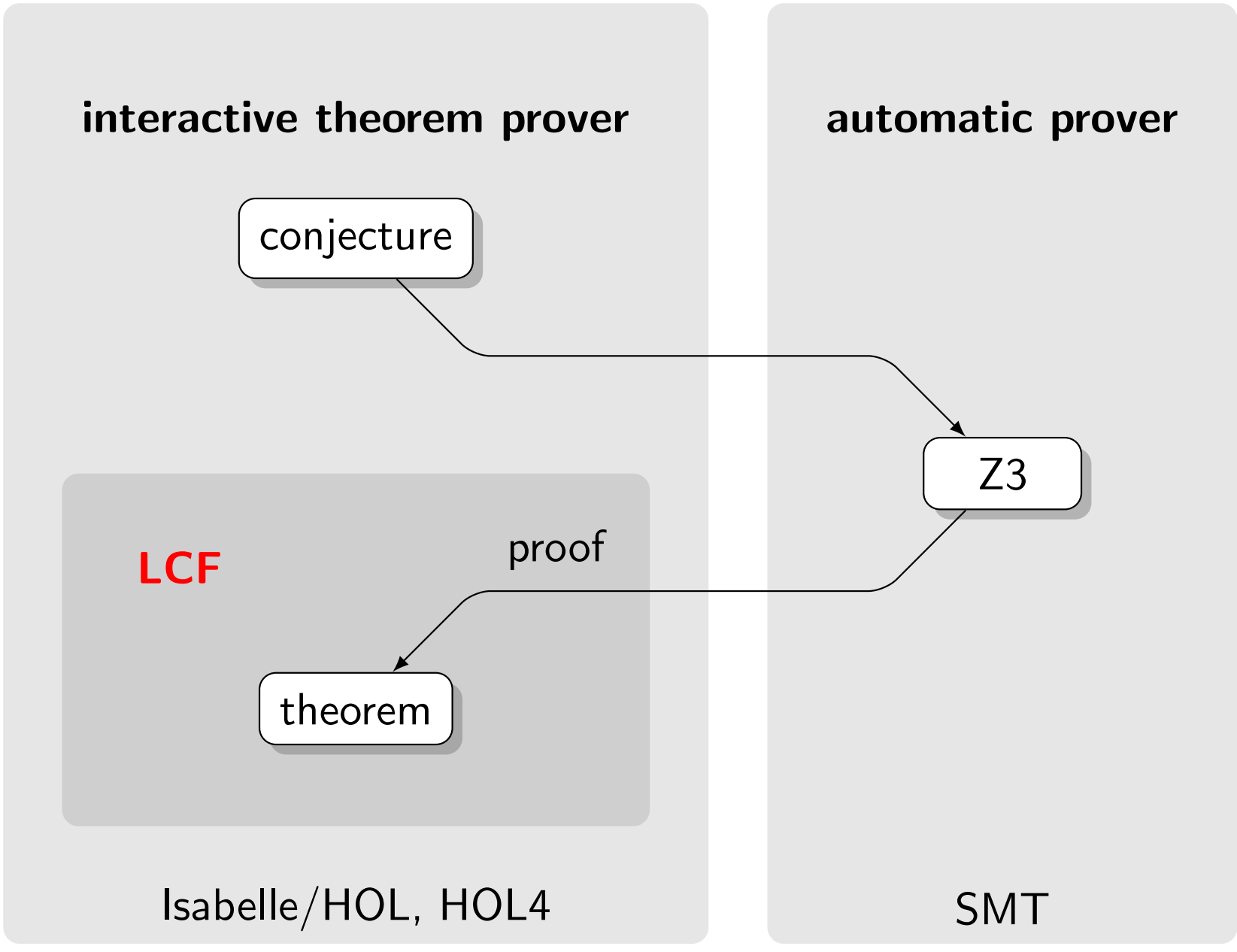
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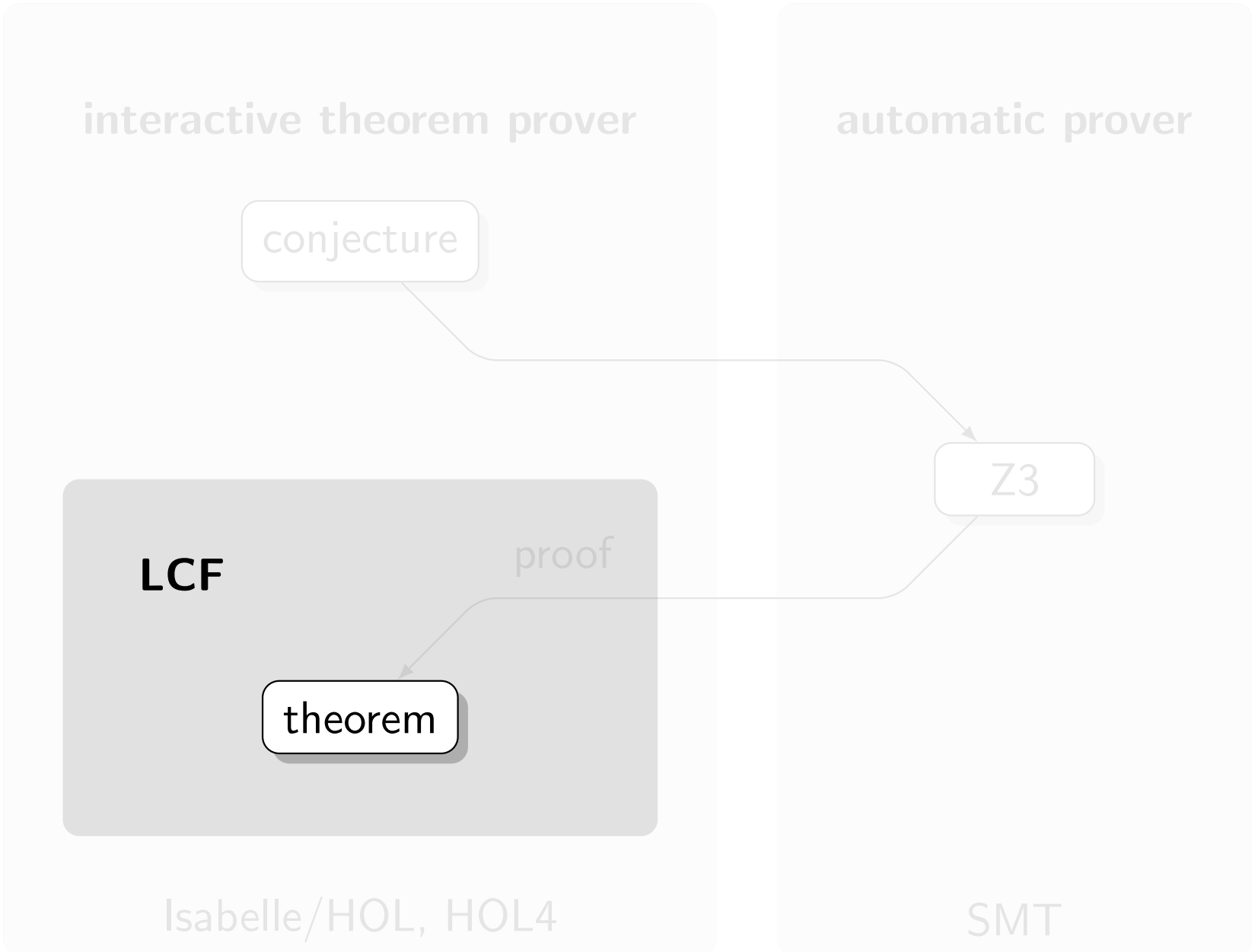
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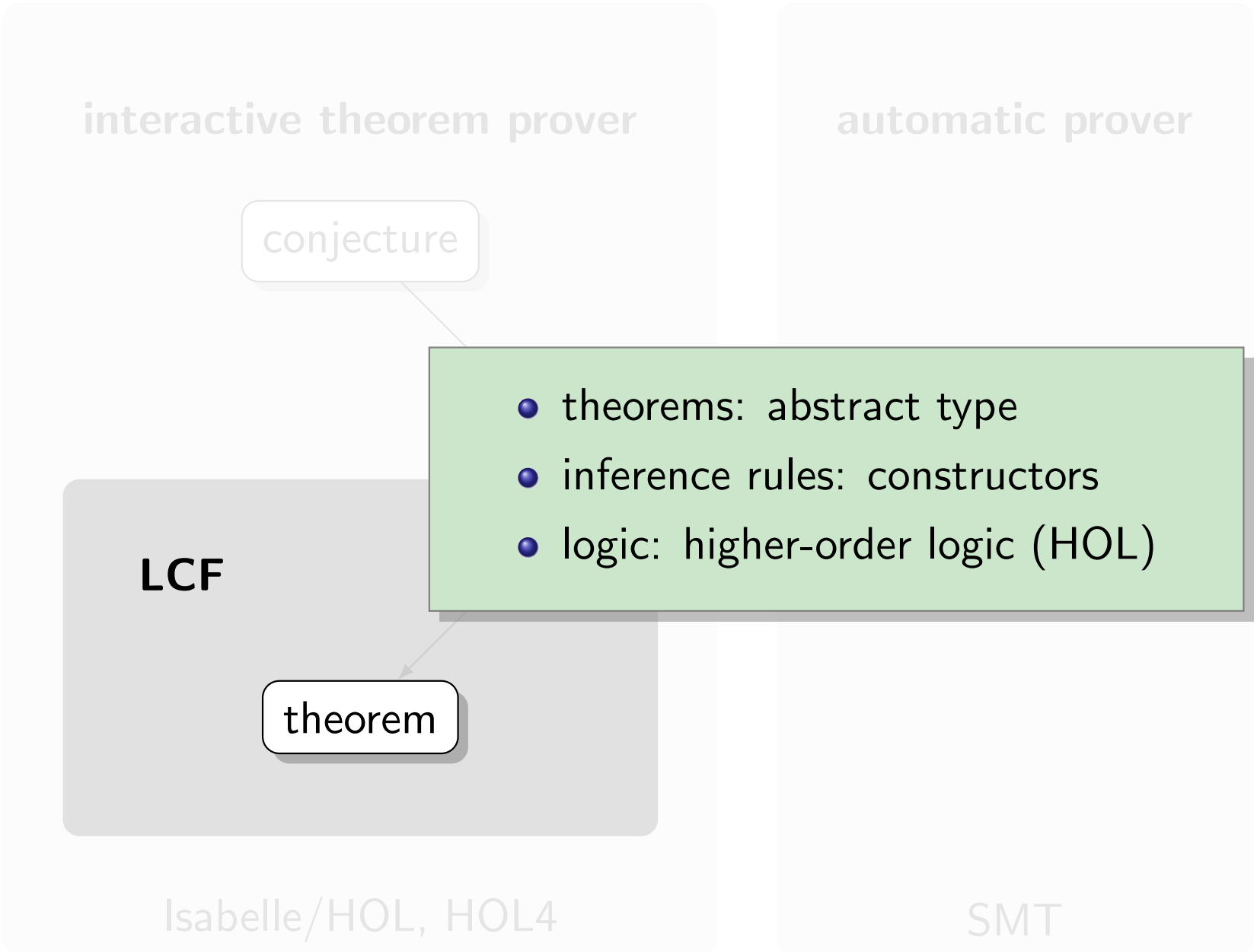
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logic: higher-order logic (HOL)
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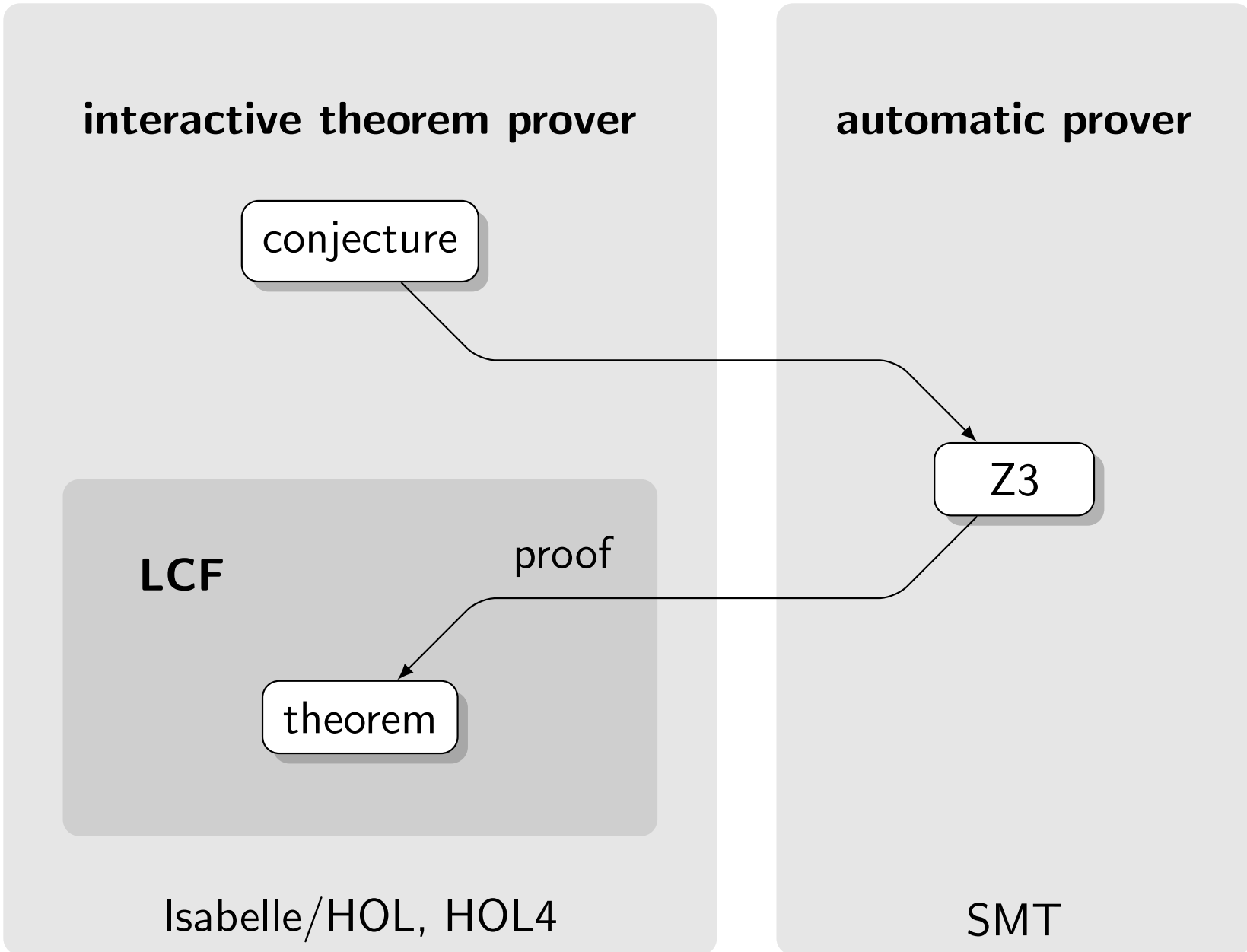
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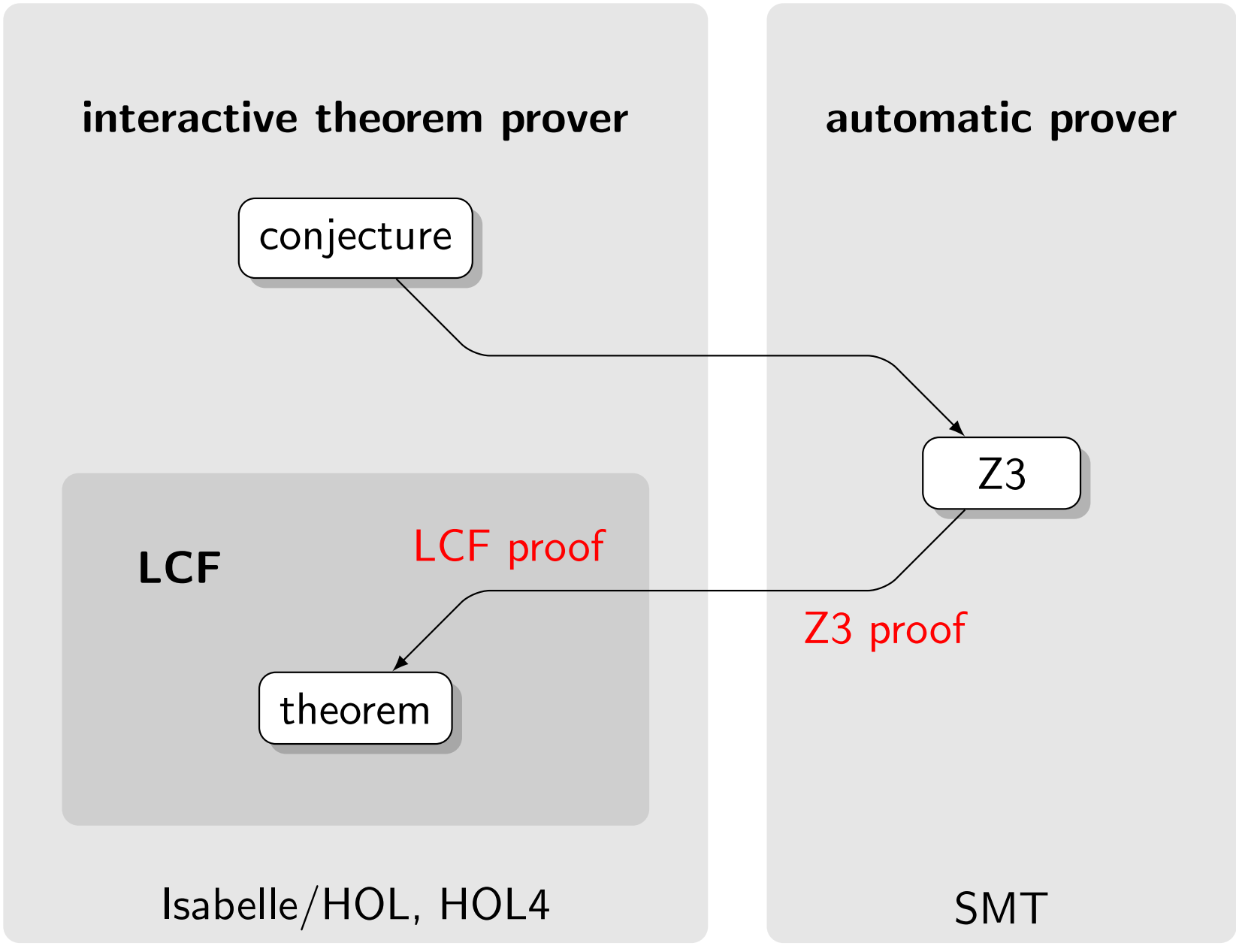
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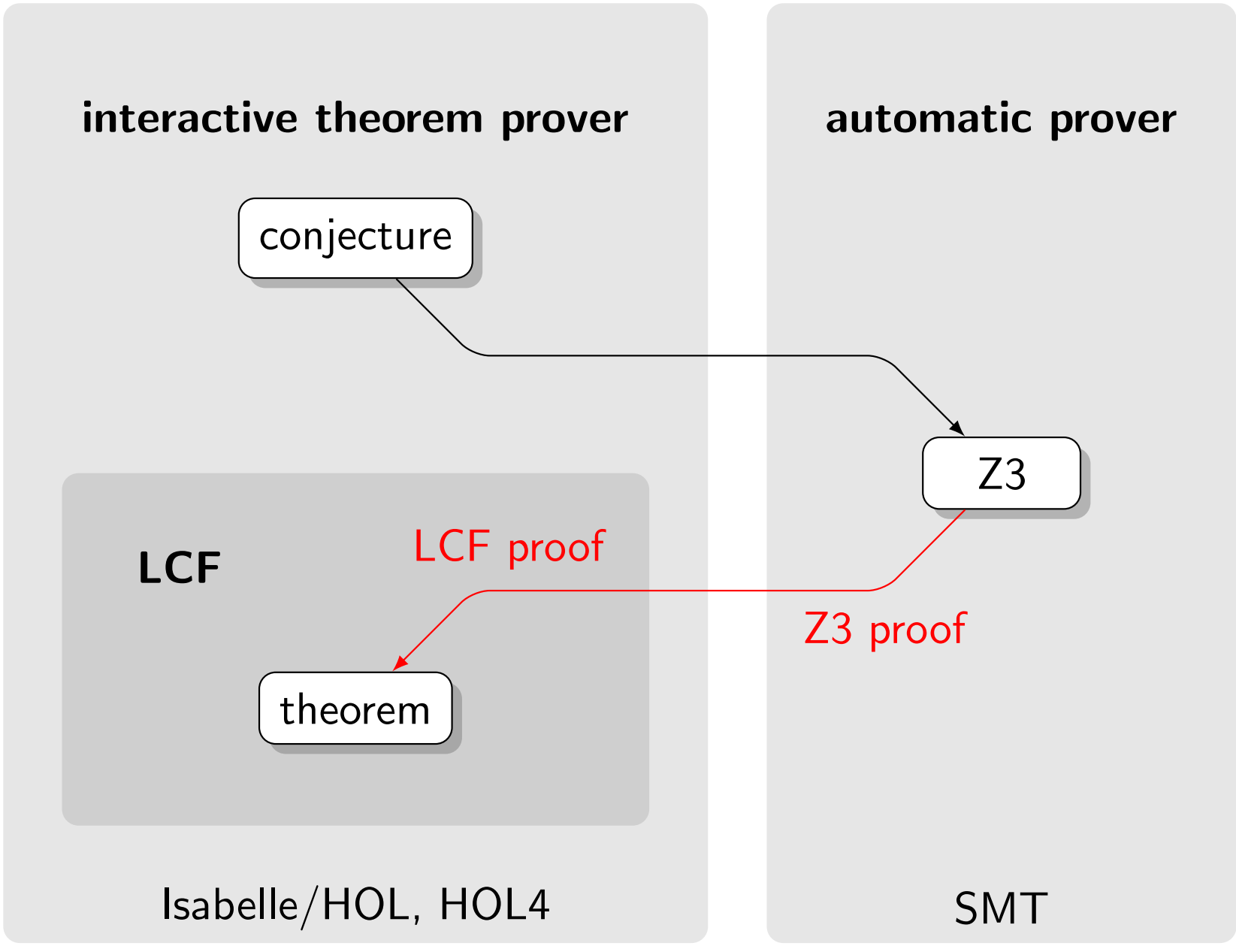
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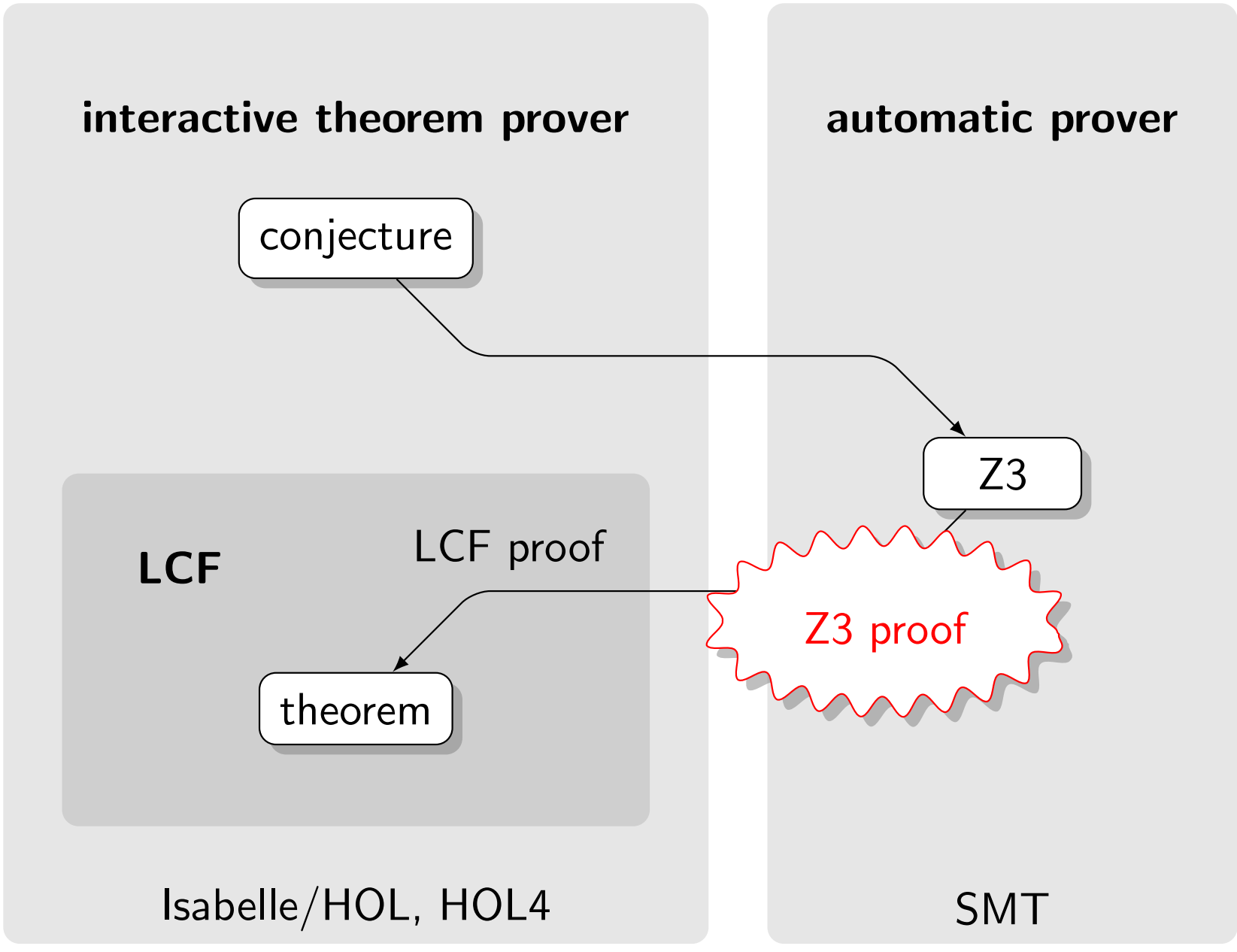
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# Z3 Proofs

Natural deduction proofs:

## Example

$$\dfrac{\dfrac{}{\neg \mathsf{True} \vdash \neg \mathsf{True}}\ \textsc{asserted} \qquad \dfrac{}{\vdash \neg \mathsf{True} \longleftrightarrow \mathsf{False}}\ \textsc{rewrite}}{\neg \mathsf{True} \vdash \mathsf{False}}\ \textsc{mp}$$

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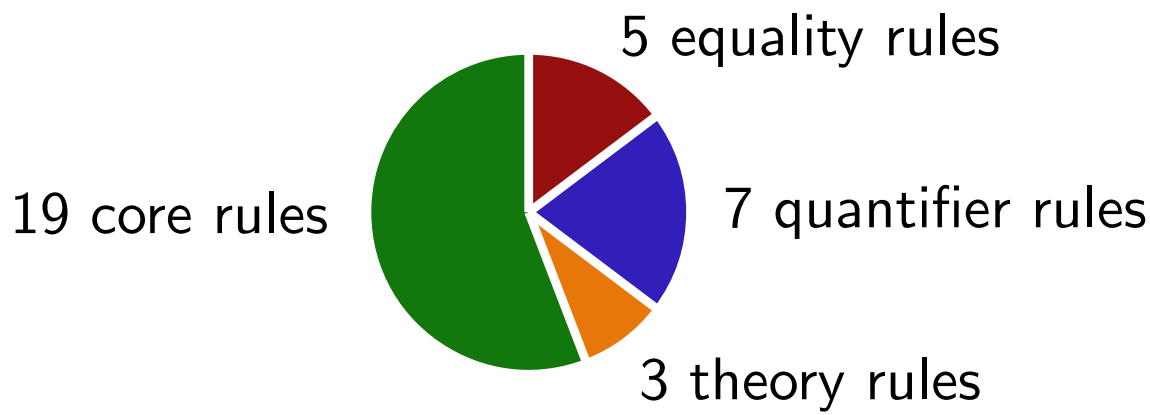
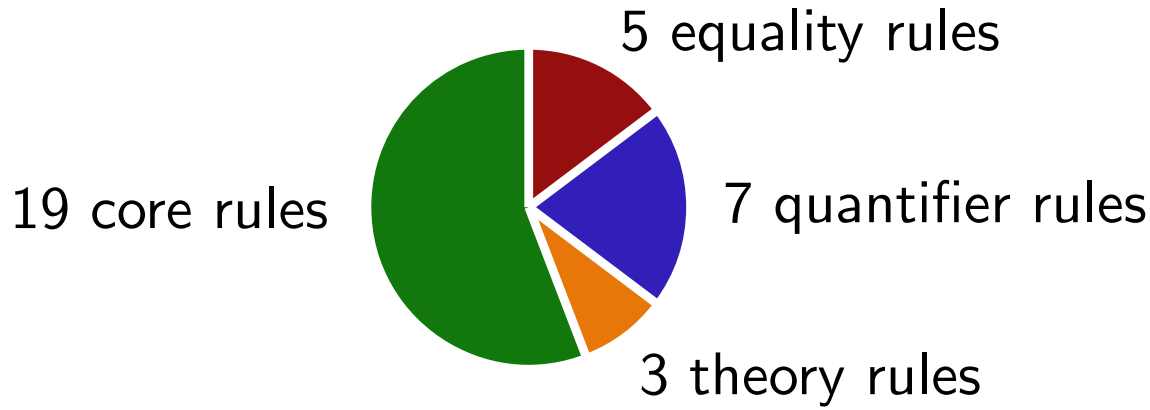
34 axiom schemata and inference rules:

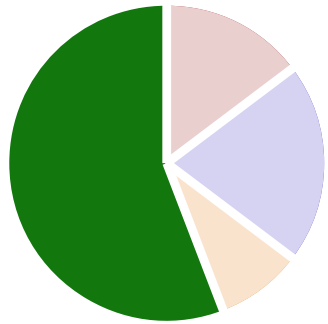


# Z3 Inference Rules



# Z3 Inference Rules

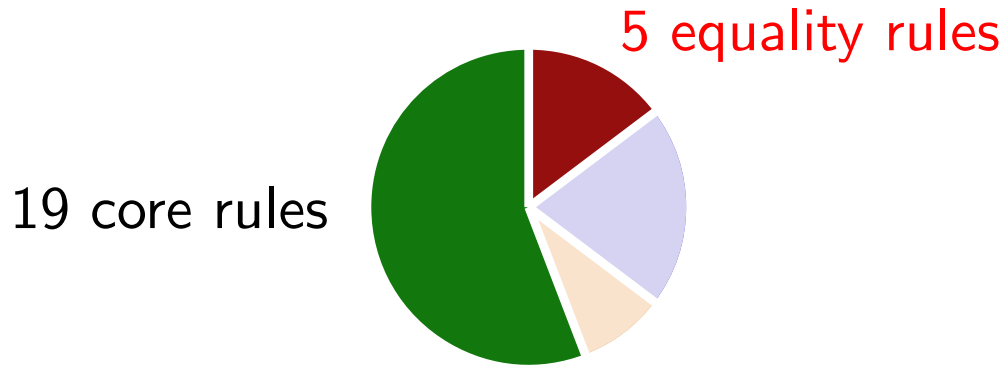
19 core rules



Core rules ASSERTED, MP, UNIT-RESOLUTION, ...

- mostly propositional reasoning

# Z3 Inference Rules

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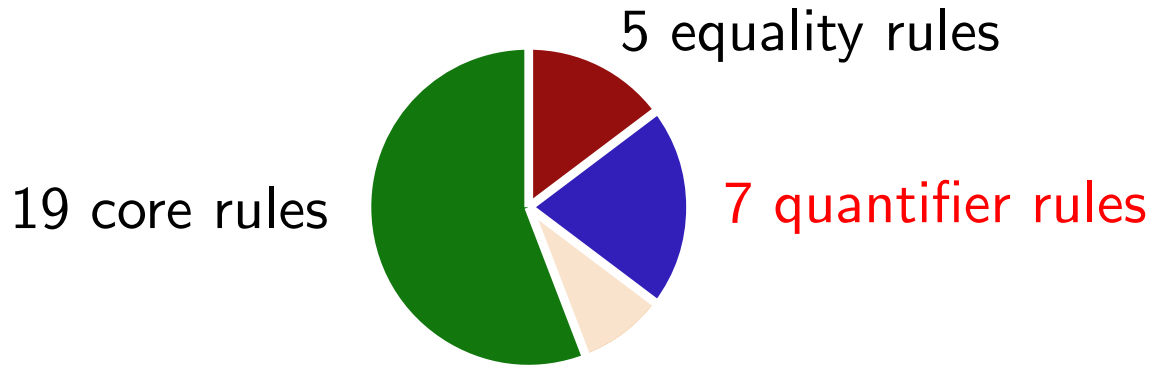


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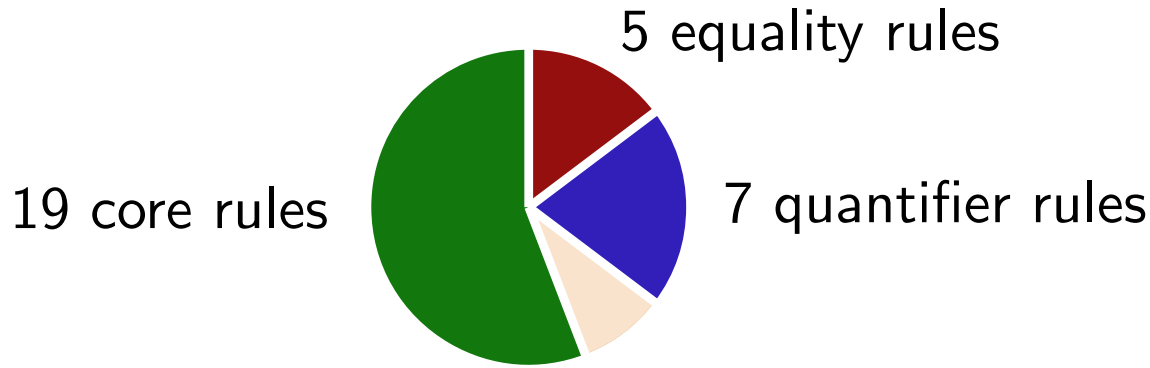


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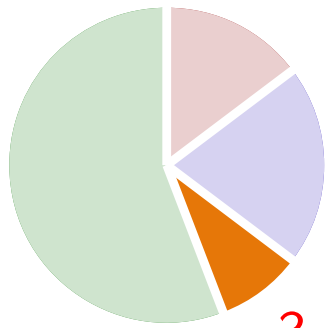
Structure of the proof, simple semantics

# Z3 Inference Rules



3 theory rules

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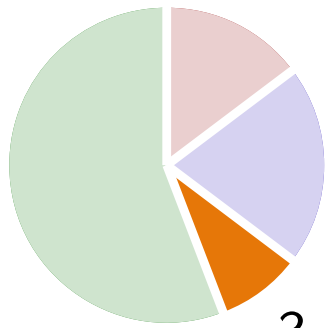
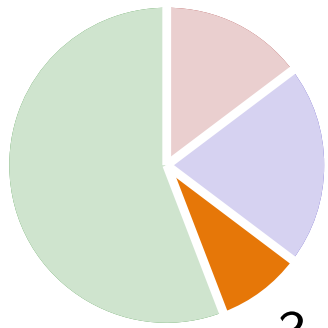



REWRITE: simplification rules

**Example**

$$\frac{}{(P \longrightarrow Q) \longleftrightarrow (Q \vee \neg P)} \qquad \frac{}{0 + x = x}$$

# Z3 Inference Rules

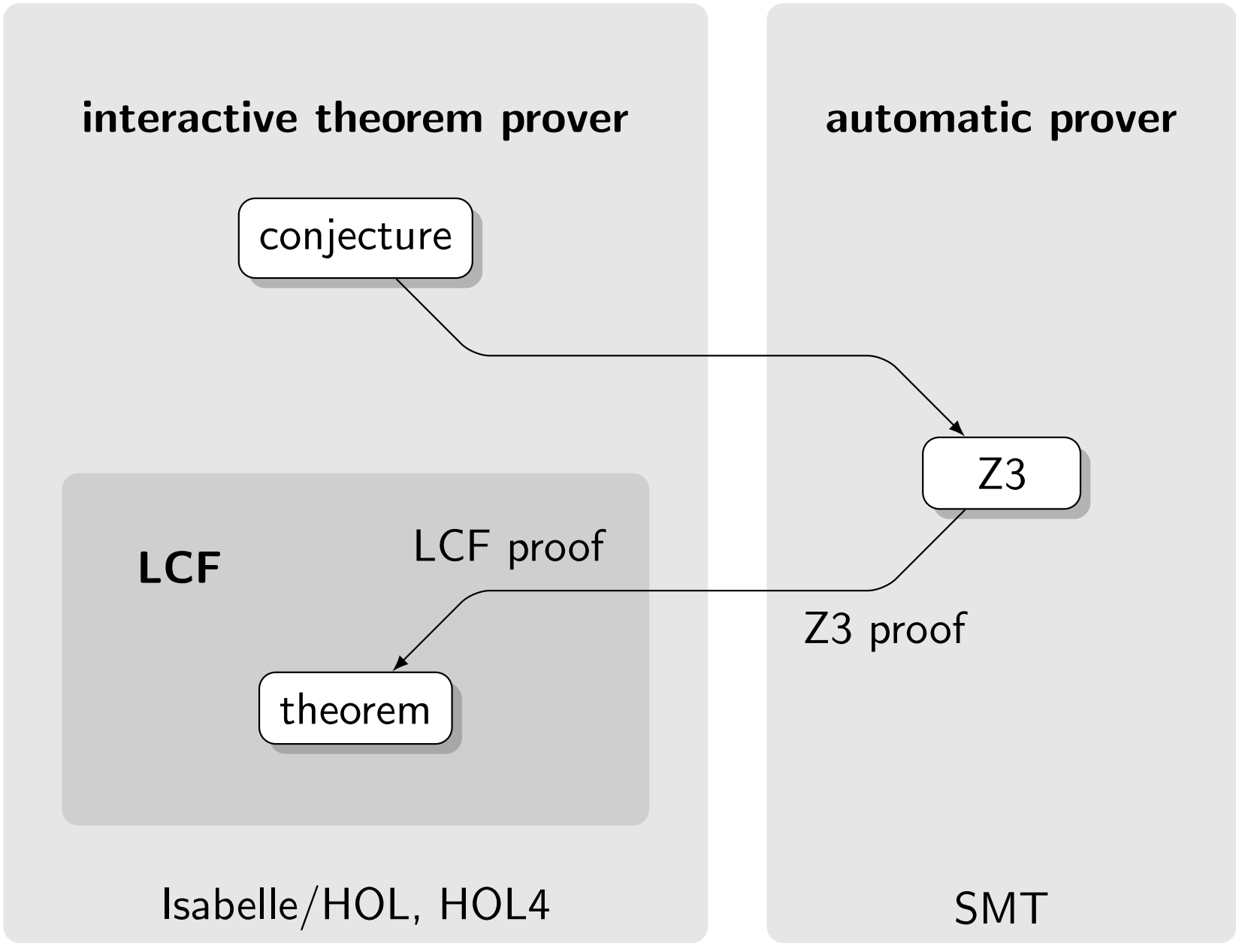


3 theory rules

TH-LEMMA: inconsistency of theory atoms

**Example**

$$\frac{0 < x - 3y \ \wedge \ x - 3y < 0}{\text{False}}$$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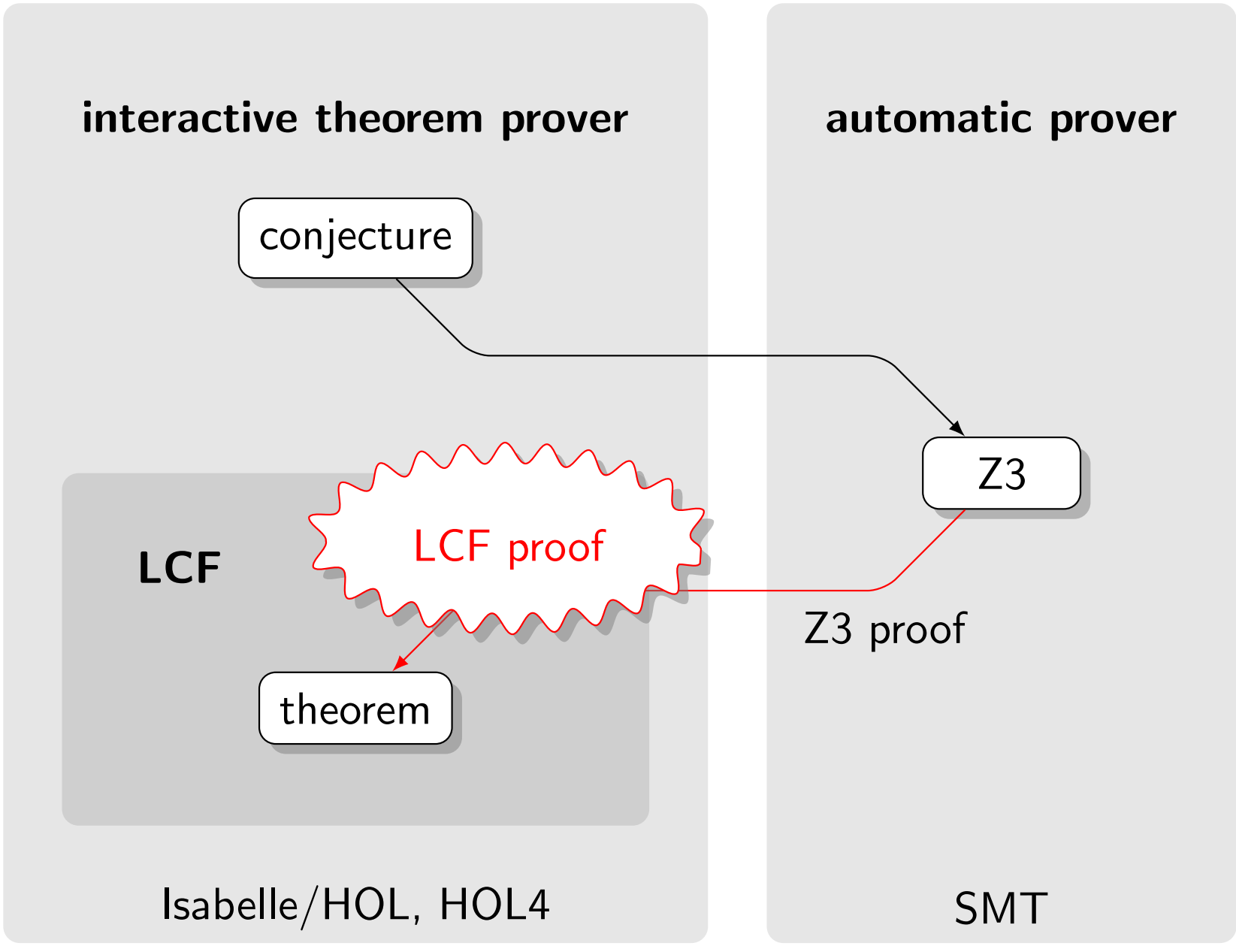
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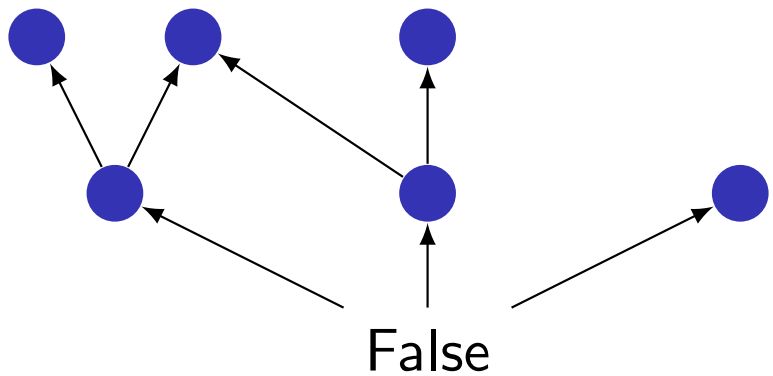
# Proof Reconstruction

Overall approach:

- one proof method for every Z3 inference rule
- depth-first traversal of Z3 proof



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Apply existing automated proof tools:

- simp, blast, auto, metis, ...

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Avoiding automated proof tools:

Speed-up of up to 3 orders of magnitude!

# Fast Proof Reconstruction

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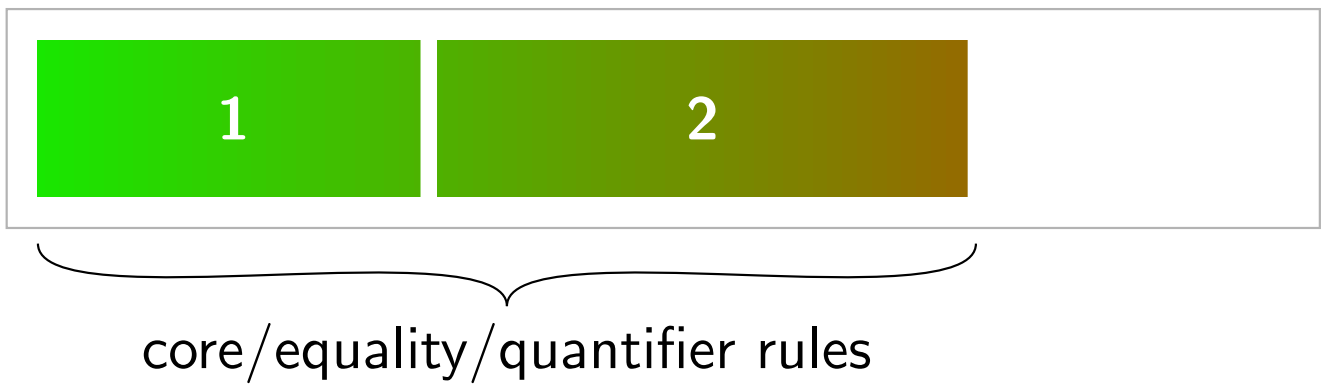

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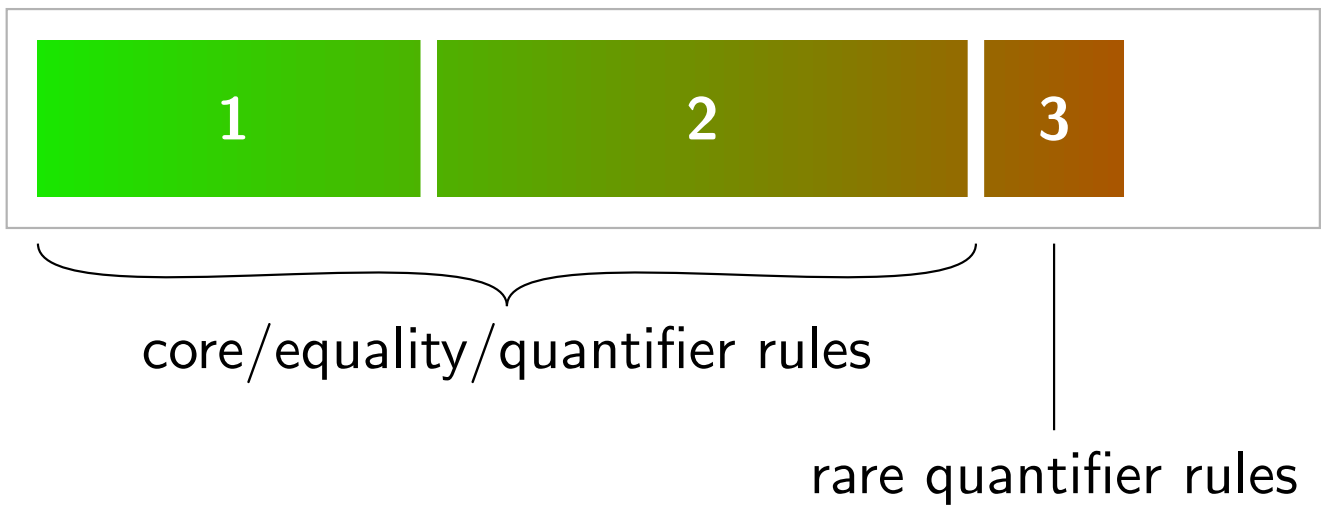

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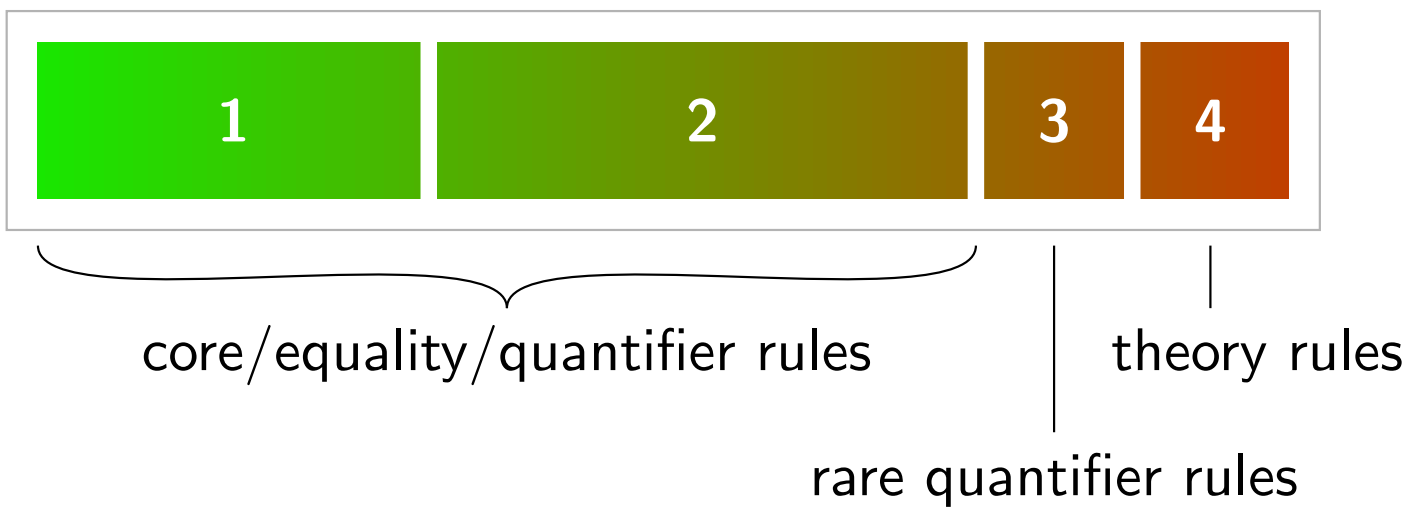
4. combinations of the above with performance optimizations



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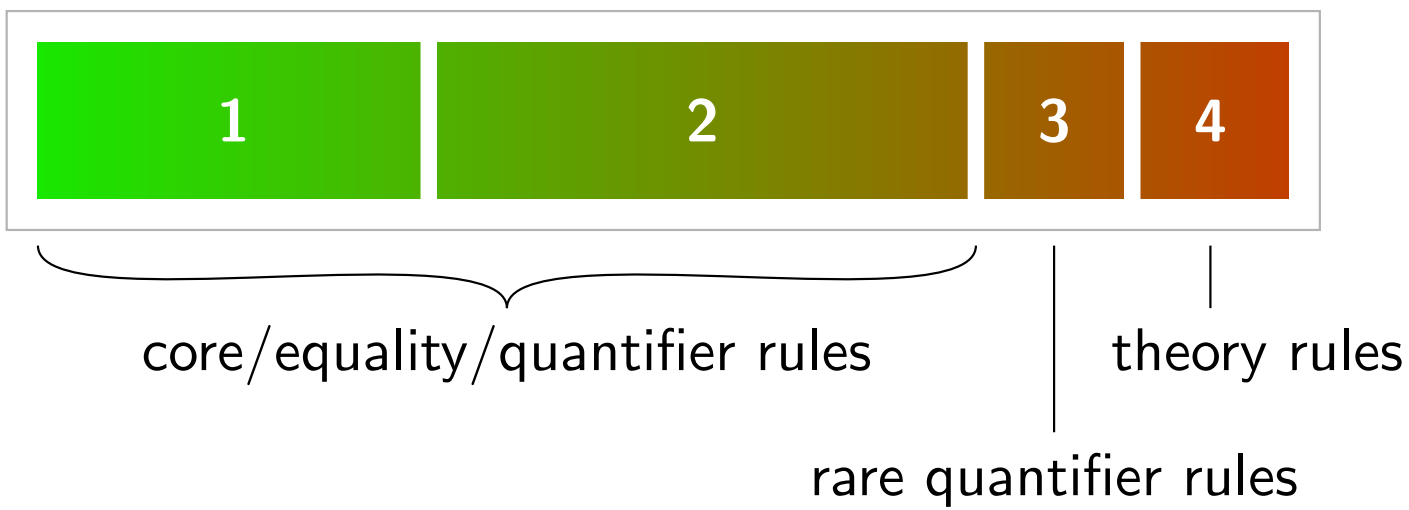

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REWRITE:

1. schematic theorems
2. conjunction/disjunction-reasoning
3. propositional reasoning
4. linear arithmetic
5. quantifier elimination for integer/real arithmetic

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## Example: REWRITE – equivalence of conjunctions

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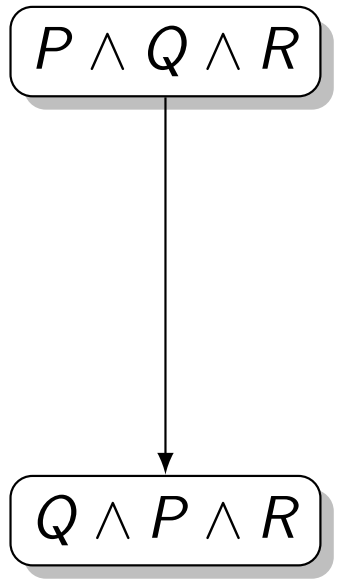
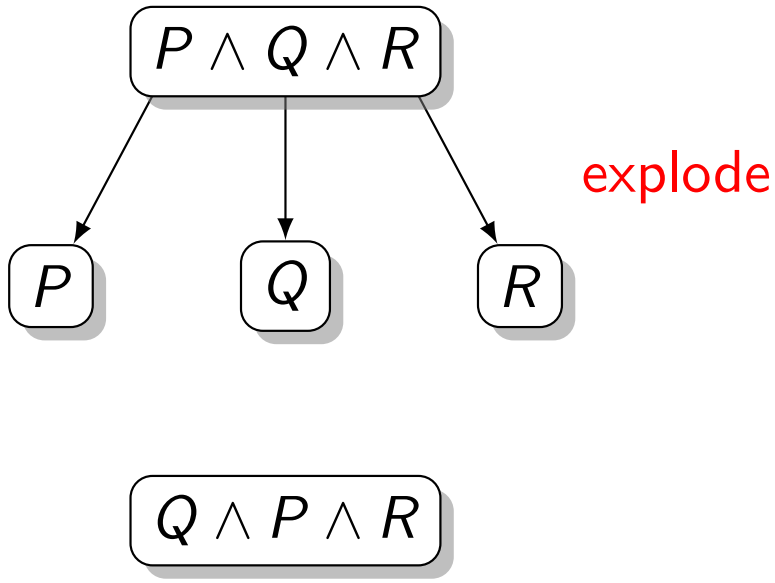


# Specific Proof Tools

## Example: REWRITE – equivalence of conjunctions

$$P \wedge Q \wedge R \longleftrightarrow Q \wedge P \wedge R$$



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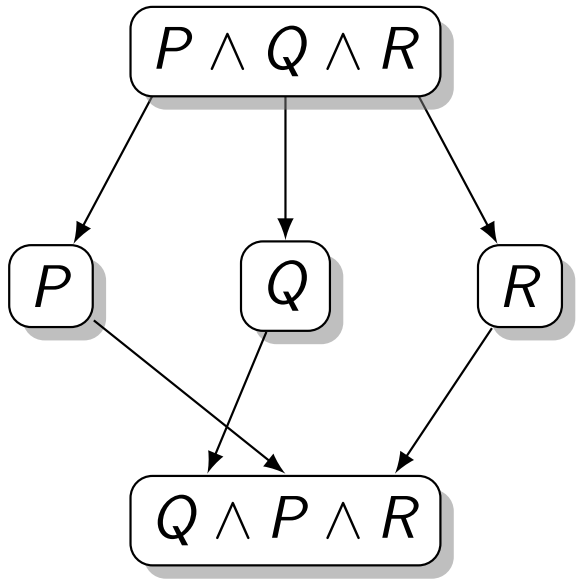


join

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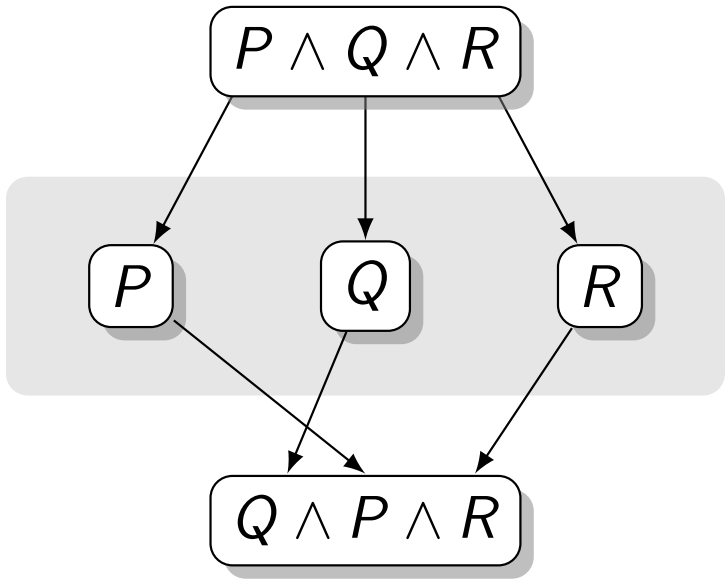

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## Example: UNIT-RESOLUTION

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Generalization:

- replace complex expressions by fresh variables
- avoids expensive preprocessing in the decision procedures
- enables shallow reasoning, more potential for theorem re-use

# Evaluation

| Logic | Z3 | | | Isabelle | | Rates | |
|---|---|---|---|---|---|---|---|
| | # | Med. Time | Med. Size | # | Med. Time | Succ. | Time-out |
| AUFLIA+p | 187 | 0.03 s | 5 KB | 187 | 0.06 s | 100% | 0% |
| AUFLIA−p | 192 | 0.04 s | 4 KB | 190 | 0.06 s | 98% | 0% |
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| QF_UFIDL | 55 | 0.30 s | 2 MB | 8 | 3.59 s | 14% | 85% |
| QF_UFLIA | 91 | 0.02 s | 107 KB | 85 | 2.12 s | 93% | 6% |
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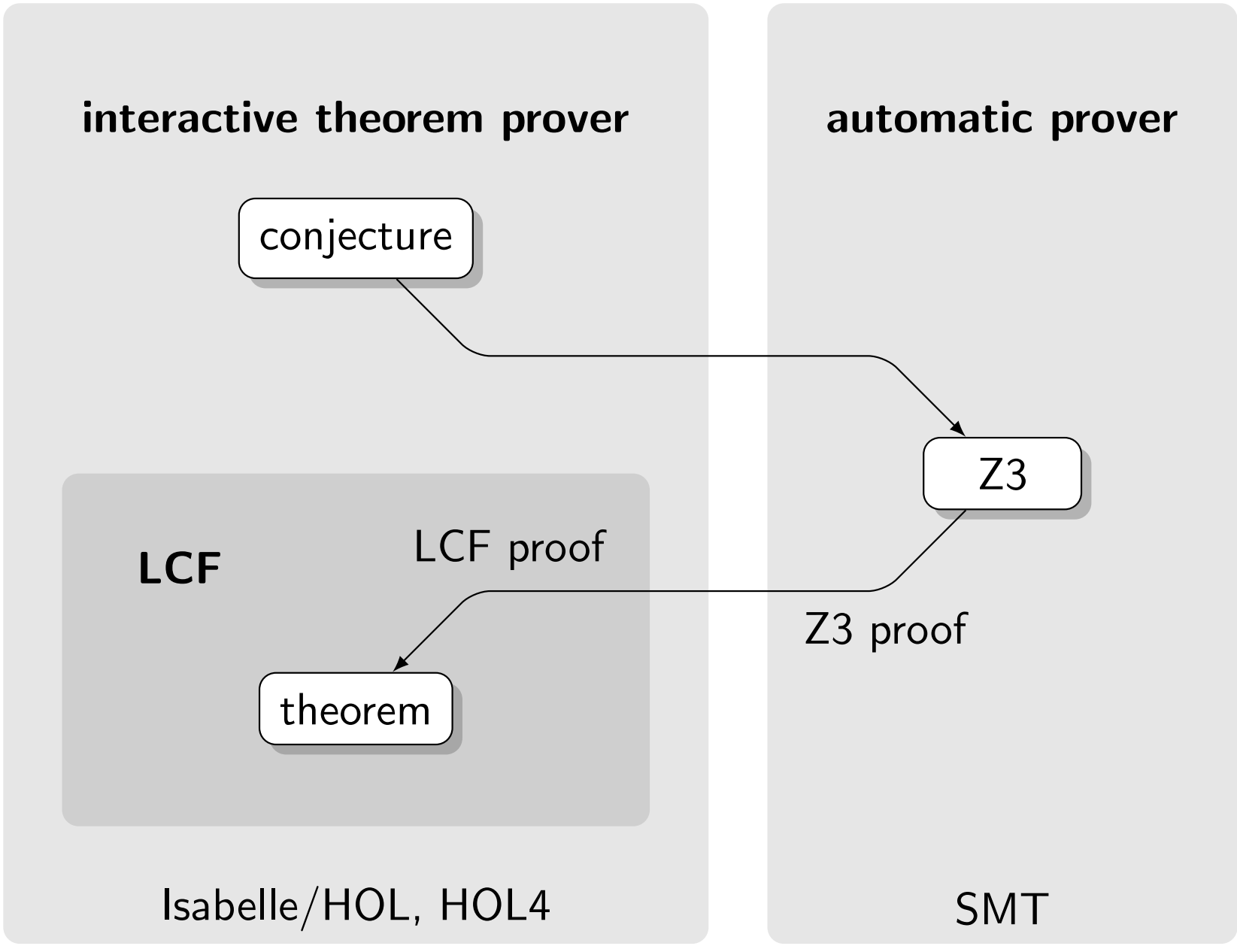
Profiling:

- highly optimized implementation
- bottleneck: theory reasoning requires expensive proof search
- 50% of the runtime is spent on 15% of all Z3 proof steps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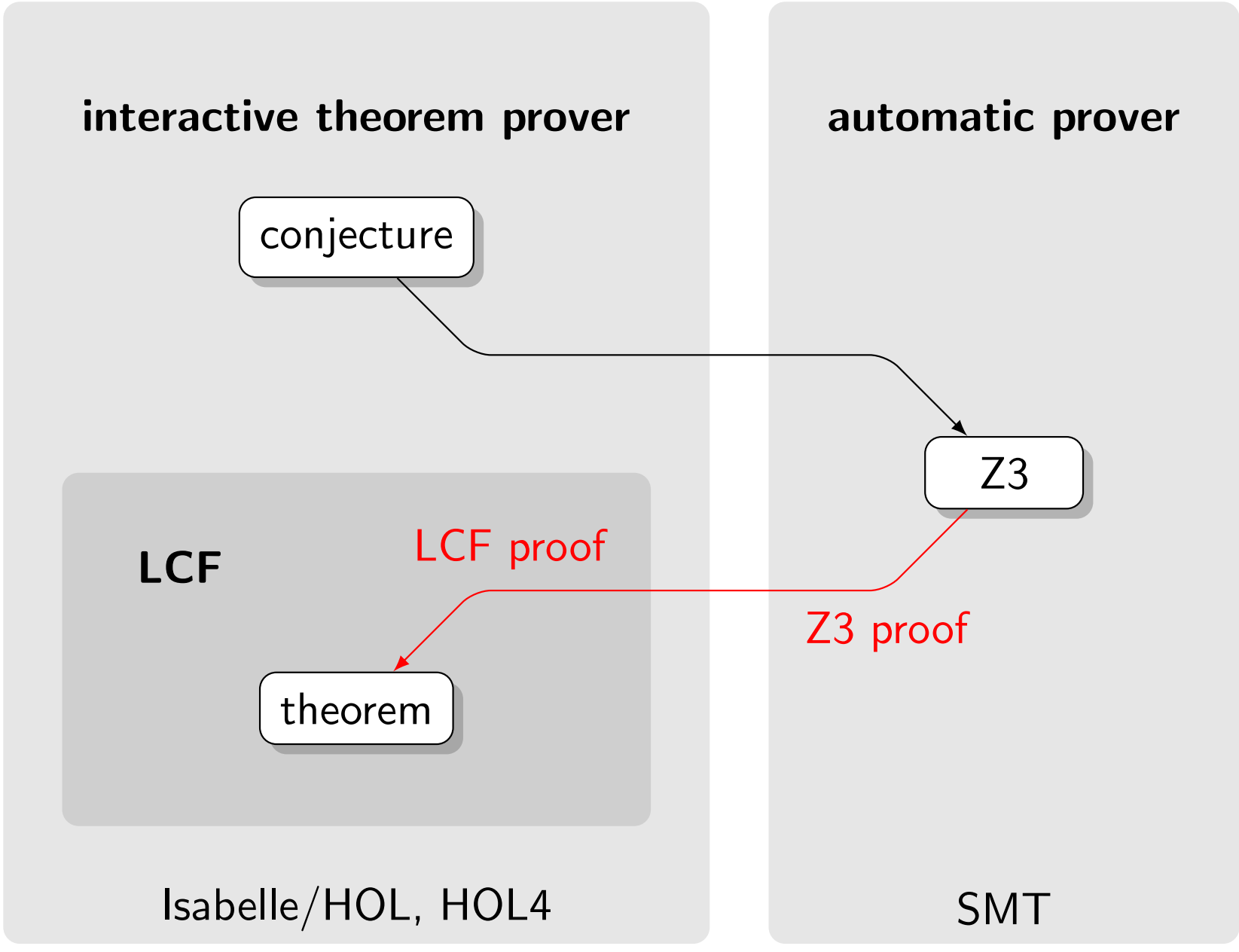
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# Conclusion

LCF-style proof reconstruction for Z3:

- highly optimized
- feasible and efficient
- slower than proof search in Z3
- Z3's proofs could be easier to check (costly theory reasoning)

Benefits for Isabelle/HOL and HOL4:

- powerful automation

Benefits for Z3:

- proof checker
- identification of several bugs

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## Soundness bug

$$(\forall\, x^T.\, x = c^T)\ \wedge\ a^T \neq b^T$$