# Using a First Order Logic to Verify That Some Set of Reals Has No Lesbegue Measure

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# Overview

- Mathematical background
- Vitali's construction of a non (Lebesgue) measurable set
- Key elements in the proof
- Formalization in ACL2(r)

# Measuring Sets of Reals

The measure (size) of a bounded interval of reals $m([a, b])$ is defined as $b - a$.

How can the notion of measure be extended to sets other than bounded intervals?

# Lebesgue

**Henri Lebesgue** defined a measure for many sets of reals.

His key idea was to cover a set of reals by a (possibly infinite) set of intervals.

But does Lebesgue's measure work for all sets of reals?

# Vitali

**Guiseppe Vitali** demonstrated that some sets cannot have a Lebesgue measure.

In fact, no “reasonable” measure can be defined over all sets of reals!

# Reasonable Measures

For all $a, b, c \in \mathbb{R}$ and $S, S_i \subset \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$,

- $m(S) \in \overline{\mathbb{R}}$
- $m(S) \geq 0$
- $m([a, b]) = b - a$
- $m(\{x + c \mid x \in S\}) = m(S)$
- $m(\bigcup_{i=1}^{\infty} S_i) = \sum_{i=0}^{\infty} m(S_i)$, for mutually disjoint $S_i$

Vitali's theorem shows no such measure can be defined over all sets of reals!

His proof uses the Axiom of Choice (and led to some controversy).

# Sidebar: The Axiom of Choice

## Axiom (of Choice)

*Suppose* $\{S_a \mid a \in \mathcal{A}\}$ *is a family of nonempty sets. Then there is a function* $c$ *such that* $c(S_a) \in S_a$.

The function $c(S_a)$ "chooses" an element from $S_a$.

This axiom is obviously true for finite families of sets.

But how do you choose an element from each subset of the reals?

# Vitali's Set V

**Definition**

Let $\approx$ be the equivalence relation defined by

$$x \approx y \Leftrightarrow x, y \in [0, 1) \wedge x - y \in \mathbb{Q}.$$

By the Axiom of Choice, there is a set $V$ that contains exactly one element from each equivalence class.

That is, $x \approx y$ if and only if $x - y$ is rational.

There are an uncountable number of equivalent classes.

The Axiom of Choice "chooses" one element from each class.

# Vitali's Proof (Outline)

The set $V \subseteq [0,1)$ is constructed so that for each $x \in [0,1)$ there is a *unique* $v \in V$ and a unique $q \in \mathbb{Q}$ such that $x = v + q$.

If $m(V) = 0$, then $m([0,1)) = 0$,
because $[0,1) \subseteq \bigcup_{q \in \mathbb{Q}} \{v + q \mid v \in V\}$.

If $m(V) > 0$, then $m([0,2)) = +\infty$,
because $\bigcup_{q \in \mathbb{Q} \cap [0,1)} \{v + q \mid v \in V\} \subseteq [0,2)$.

So if $V$ has a measure (no matter what it is), and if the measure is countably additive and translation invariant, the measure does not yield the correct result for intervals.

# To Formalize the Proof

The proof is based on three pillars:

1. Properties of the real numbers
2. Properties of sets of real numbers, including countable unions
3. The Axiom of Choice

ACL2 is a first-order logic with minimal support for quantifiers and with only finite data structures.

The main contribution of this work was demonstrating that Vitali's theorem can be formalized in this logic.

# The Real Numbers

ACL2 has support for the rationals, not the real numbers.

ACL2(r) is a variant of ACL2 that supports the reals using nonstandard analysis.

In nonstandard analysis, there are real numbers that behave like "infinitesimals" and "infinite" quantities.

This is exactly what Newton had in mind when he was thinking about calculus!

Think of the $\epsilon$ or $\Delta x$ as infinitesimals. Then $1/\epsilon$ and $1/\Delta x$ are infinite "hyperreals".

# The Transfer Principle

This plays an important role in nonstandard analysis.

Any first-order, classical property that is true of all standard reals must also be true of all reals (including the hyperreals).

We use this principle extensively to reason about infinite sums and infinite unions.

# Sets of Real Numbers

We need a *finite* representation of sets of real numbers.

We chose to represent sets by unary functions that recognize their element. E.g., the set of even numbers is represented by the function $f(x) \equiv (\exists y)(y \in \mathbb{Z} \wedge 2y = x)$.

Since ACL2 is first-order, it cannot reason about functions! Rather, we reason about the *term* $(\exists y)(y \in \mathbb{Z} \wedge 2y = x)$. The term is really just a collection of symbols.

To reason about terms, we implemented an interpreter over terms (to determine set membership), as well as functions that manipulate terms (for the other set operations).

# Just Enough Sets

The sets we can construct in our theory are not (nearly) exhaustive.

They do not, for example, form a $\sigma$-algebra.

Moreover, the interpreter needs to know *a priori* all functions that may be used to define sets.

Among these functions we include the equivalence function defined by Vitali.

# $x \in S$?

Suppose $S$ is a set in our formalization.

That is, $S$ is a term that corresponds to a unary function. (It's a $\lambda$-expression.)

Then $x \in S$ if and only if $eval(S, x)$ is true.

Here, *eval* is our evaluator over terms.

# $S_1 \cup S_2$

Suppose *S1* and *S2* are sets in our formalization.

That is, *S1* and *S2* are terms that corresponds to unary functions.

Then their union *S* is a term that corresponds to the unary function "*S1* or *S2*".

This can be computed mechanically, without any understanding of the sets *S1* and *S2*.

# $S_1 \cup S_2 \cup S_3 \cup \ldots$

First of all, ACL2 only allows finite terms, so it cannot represent the infinite family $S1, S2, S3, \ldots$.

We represent this family with a function $F$ such that $F(i) = Si$.

Note that $F$ is really a term, not a function because ACL2 is first order. The interpreter provides the semantics, so we really mean $eval(F, i)$ instead of $F(i)$!

Note also that $F(i)$ is not a set. It, too, is a term!

## $S_1 \cup S_2 \cup S_3 \cup \ldots$

Recall that $F$ is a term and that $eval(F, i)$ is a term that represents $Si$.

Define $U(F, n)$ to be the term that represents $S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n$.

$U(F, n)$ is a simple recursive function.

Consider $S = U(F, N)$ where $N$ is a fixed (but arbitrary) "infinite" hyperreal.

Using the Transfer Principle, we can show that $x \in S$ if and only if $x \in S_1 \cup S_2 \cup S_3 \cup \ldots$, for any real (but not necessarily hyperreal) number $x$.

# The Axiom of Choice

ACL2 does not have a direct equivalent of the Axiom of Choice.

It does support the introduction of Skolem functions, which are similar.

This defines a “square root” function, for example:

(**defchoose** *sqrt* (*y*) (*x*)
  (*equal* ($*$ *y* *y*) *x*))

The defining axiom states that **if** there is a $y$ such that $x = y^2$, **then** (*sqrt* $x$) also has this property.

# Choice and Equivalent Classes

An equivalence relation automatically defines a set of equivalence classes.

We are often interested in choosing a canonical element from each class.

For example, if the equivalence relation is "equal, mod 3", we have three equivalence classes: $\{0, 3, 6, \dots\}$, $\{1, 4, 7, \dots\}$, and $\{2, 5, 8, \dots\}$

The canonical elements may be $[0] = 0$, $[1] = 1$, $[2] = 2$.

The important fact is that $[0] = [3] = [6] = \cdots = 0$.

# Defchoose and Equivalent Classes

Using **defchoose** does not guarantee that the element chosen from each equivalent class is canonical.

```
(defchoose elem (y) (x)
  (and (integerp y)
       (≥ y 0)
       (= (remainder (− y x) 3)
          0)))
```

The problem is that (*elem* 0) is not necessarily equal to (*elem* 3). They are just guaranteed to be equivalent.

This chooses an equivalent element for each element, not for the whole class! So it is not a choice function.

# Strong Defchoose and Equivalent Classes

Recently, ACL2 introduced the idea of "strong defchoose" which **does** guarantee that it picks a canonical element from each equivalence class.

```
(defchoose elem (y) (x)
  (and (integerp y)
       (≥ y 0)
       (= (remainder (− y x) 3)
          0))
  :strengthen t)
```

With the *:strengthen* option, **defchoose** will select the same $y$ for each $x$ in an equivalence class.

# Future Work

There are many wonderful consequences of the Axiom of Choice. Here's a similar result:

**Theorem (Banach-Tarski Paradox)**

*A ball can be decomposed into a finite number of point sets and reassembled (using only rigid, geometrical transformations, such as translations and rotations) into two balls identical to the original.*

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All images courtesy of Wikipedia.