Differentiation Under the Integral Sign

We often need to differentiate under the integral sign:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_X f(x,t)\,\mathrm{d}\mu(x) = \int_X \partial_t f(x,t)\,\mathrm{d}\mu(x),$$

which allows us to exchange the order of derivative and integral sign.

The following is a rigorous statement of the result using Fubini's theorem.

Theorem 1. Let (X,μ) be a σ-finite measure space and $I=[a,b]\subset\mathbb{R}$ a compact interval. Let $f: X\times I\to\mathbb{R}$ be a measurable function satisfying:

1. *Absolute Continuity:* For μ-a.e. $x\in X$, the map $t\mapsto f(x,t)$ is absolutely continuous on I.
2. *Integrable Function:* There exists $t_0\in I$ such that $f(\cdot,t_0)$ is integrable on X:
$$\int_X |f(x,t_0)|\,\mathrm{d}\mu(x) < \infty.$$
3. *Integrable Derivative:* The partial derivative $\partial_t f$ is integrable on $X\times I$:
$$\int_X\int_I |\partial_t f(x,t)|\,\mathrm{d}t\,\mathrm{d}\mu(x) < \infty.$$

Then the function $F(t) := \int_X f(x,t)\,\mathrm{d}\mu(x)$ is absolutely continuous on I, and for almost every $t\in I$:

$$F'(t) = \int_X \partial_t f(x,t)\,\mathrm{d}\mu(x).$$

Proof. 1. Integral representation of f. Since $t\mapsto f(x,t)$ is absolutely continuous for almost every x (Assumption 1), the Fundamental Theorem of Calculus implies that for any $t\in I$:

$$f(x,t) = f(x,t_0) + \int_{t_0}^{t}\partial_t f(x,s)\,\mathrm{d}s \quad \text{for } \mu\text{-a.e. } x.$$

2. Finiteness of $F(t)$. To ensure $F(t)$ is well-defined, we check the integrability of $f(\cdot,t)$. By the triangle inequality:

$$|f(x,t)| \le |f(x,t_0)| + \left|\int_{t_0}^{t}\partial_t f(x,s)\,\mathrm{d}s\right| \le |f(x,t_0)| + \int_I |\partial_t f(x,s)|\,\mathrm{d}s.$$

Integrating over X:

$$\int_X |f(x,t)|\,\mathrm{d}\mu(x) \le \int_X |f(x,t_0)|\,\mathrm{d}\mu(x) + \int_X \int_I |\partial_t f(x,s)|\,\mathrm{d}s\,\mathrm{d}\mu(x).$$

Thus, $f(\cdot,t) \in L^1(\mu)$ for all $t \in I$, so $F(t)$ is finite everywhere.

3. Application of Fubini's theorem. Consider the difference $F(t) - F(t_0) = \int_X \left(\int_{t_0}^t \partial_t f(x,s)\,\mathrm{d}s\right)\mathrm{d}\mu(x)$. Let $h(x,s) := \partial_t f(x,s)\cdot \mathbf{1}_{[t_0,t]}(s)$. Since $|\partial_t f| \in L^1(X\times I)$, Fubini's Theorem applies:

$$\int_X \int_{t_0}^t \partial_t f(x,s)\,\mathrm{d}s\,\mathrm{d}\mu(x) = \int_{t_0}^t \left(\int_X \partial_t f(x,s)\,\mathrm{d}\mu(x)\right)\mathrm{d}s.$$

4. Conclusion. Let $G(s) := \int_X \partial_t f(x,s)\,\mathrm{d}\mu(x)$. Then $F(t) = F(t_0) + \int_{t_0}^t G(s)\,\mathrm{d}s$. By fundamental theorem of calculus for Lebesgue integrals, F is absolutely continuous and $F'(t) = G(t)$ for a.e. t. □

Remark 1. A common sufficient condition (stronger than Assumption 3) is to start from the difference quotient $\frac{f(x,t+\varepsilon)-f(x,t)}{\varepsilon}$ and justify exchanging $\lim_{\varepsilon\to 0}$ and $\int_X(\cdot)\,\mathrm{d}\mu$ by dominated convergence. This typically requires an *integrable envelope* that dominates the difference quotients (or, equivalently, dominates $|\partial_t f(x,t)|$ uniformly for t near some t_0, i.e. there exists $C \in L^1(X)$ and a neighborhood U of t_0 such that $|\partial_t f(x,t)| \le C(x)$ for all $t \in U$ and μ-a.e. x), which is more restrictive than assuming $\partial_t f \in L^1(X \times I)$ above.

Conceptually, the proof above applies the Fundamental Theorem of Calculus in the t-variable and then reduces the interchange-of-derivative question to commuting two integrals, which can be handled by Fubini's theorem.

See `https://planetmath.org/differentiationundertheintegralsign`.

Remark 2. (*Differentiating under the integral sign with a moving domain.*) If the integration region depends on a parameter, say $\Omega(t) \subset \mathbb{R}^d$, then one has an extra boundary term. For instance, assume $\partial\Omega(t)$ is smooth and evolves with a velocity field $v(\cdot,t)$, in the sense that each boundary point $x(t) \in \partial\Omega(t)$ moves according to $\dot{x}(t) = v(x(t),t)$. Let $n(x,t)$ denote the outward unit normal to $\partial\Omega(t)$ at $x \in \partial\Omega(t)$, so the normal speed is $v(x,t)\cdot n(x,t)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} f(x,t)\,\mathrm{d}x = \int_{\Omega(t)} \partial_t f(x,t)\,\mathrm{d}x + \int_{\partial\Omega(t)} f(x,t)\,v(x,t)\cdot n(x,t)\,\mathrm{d}S(x).$$

This is known as the *Reynolds transport theorem* (a Leibniz rule for moving domains).

1 Divergence Theorem on $\mathbb{R}^d$

We use this to prove the divergence theorem on $\mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \nabla f(x)\,\mathrm{d}x = 0, \quad \forall f \in W^{1,1}(\mathbb{R}^d),$$

where $W^{1,1}(\mathbb{R}^d)$ is the Sobolev space of weakly differentiable functions with $f \in L^1(\mathbb{R}^d)$ and $\nabla f \in L^1(\mathbb{R}^d; \mathbb{R}^d)$.

We first establish the result for functions that satisfy the *Absolute Continuity on Lines* (ACL) property.

Theorem 2 (Divergence Theorem for ACL Functions)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that:

1. $f \in L^1(\mathbb{R}^d)$.
2. For each $i \in \{1, \dots, d\}$, f is absolutely continuous on almost every line parallel to the i-th coordinate axis.
3. The classical partial derivatives $\partial_i f$ (which exist a.e. by condition 2) are in $L^1(\mathbb{R}^d)$.

Then for each $i \in \{1, \dots, d\}$, we have $\int_{\mathbb{R}^d} \partial_i f(x)\,\mathrm{d}x = 0$.

Proof. Fix $i \in \{1, \dots, d\}$. By the translation invariance of the Lebesgue measure, the integral of f is invariant under shifts in the e_i direction:

$$F(h) := \int_{\mathbb{R}^d} f(x + h\mathbf{e}_i)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(y)\,\mathrm{d}y = \mathrm{const}.$$

We check the conditions to differentiate $F(h)$ under the integral sign:

- *AC:* $h \mapsto f(x + h\mathbf{e}_i)$ is AC for a.e. x by assumption.
- *Integrability:* $f \in L^1$ ensures $F(h)$ is finite.
- *Derivative:* The partial derivative $\partial_h f(x + h\mathbf{e}_i) = \partial_i f(x + h\mathbf{e}_i)$ is integrable over $\mathbb{R}^d \times I$ by Tonelli's theorem: $\int_I \int_{\mathbb{R}^d} |\partial_i f(x + s\mathbf{e}_i)|\,\mathrm{d}x\,\mathrm{d}s = |I|\|\partial_i f\|_1 < \infty$.

Applying the differentiation theorem:

$$0 = F'(h) = \int_{\mathbb{R}^d} \partial_i f(x + h\mathbf{e}_i)\,\mathrm{d}x = \int_{\mathbb{R}^d} \partial_i f(x)\,\mathrm{d}x,$$

where we used the translation invariance on $\partial_i f$. □

Now we connect this result to the Sobolev space $W^{1,1}(\mathbb{R}^d)$ using the standard characterization of Sobolev functions.

Theorem 3 (ACL Characterization of $W^{1,1}$)**.** A function f belongs to $W^{1,1}(\mathbb{R}^d)$ if and only if there exists a function $\hat{f}$ such that $\hat{f} = f$ almost everywhere, $\hat{f}$ is ACL, and the classical partial derivatives $\partial_i \hat{f}$ belong to $L^1(\mathbb{R}^d)$ for all $i = 1, \dots, d$. In this case, $\partial_i \hat{f}$ coincides with the weak derivative $\partial_i f$ almost everywhere.

Theorem 4 (Divergence Theorem for $W^{1,1}$)**.** If $f \in W^{1,1}(\mathbb{R}^d)$, then $\int_{\mathbb{R}^d} \nabla f(x)\,\mathrm{d}x = 0$.

Proof. Let $\hat{f}$ be as in the ACL characterization above. Since $f \in W^{1,1}(\mathbb{R}^d)$, $\hat{f}$ satisfies all three conditions of the ACL Divergence Theorem: $\hat{f} \in L^1$, $\hat{f}$ is ACL, and $\partial_i \hat{f} = D_i f \in L^1$. Thus,

$$\int_{\mathbb{R}^d} \partial_i f(x) \,\mathrm{d}x = \int_{\mathbb{R}^d} \partial_i \hat{f}(x) \,\mathrm{d}x = 0.$$

The result for the gradient follows by applying this component-wise. $\square$