

# ARBITRARY RECURSION

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# One Recursive Call

arbitrary, possibly non-terminating, recursive "definition" (e.g. a program-mode function):

$$f(\bar{x}) \stackrel{?}{=} \text{if } a(\bar{x}) \text{ then } b(\bar{x}) \text{ else } c(\bar{x}, f(\bar{d}(\bar{x}))) \quad \bar{x} = (x_1, \dots, x_n) \quad \bar{d}(\bar{x}) = (d_1(\bar{x}), \dots, d_n(\bar{x})) \quad n > 0$$

$$a \in U^n \quad b: U^n \rightarrow U^m \quad d_i: U^n \rightarrow U \quad c: U^n \times U^m \rightarrow U^m \quad m > 0 \quad (\text{number of results})$$

$$t(\bar{x}) \triangleq [\exists k \in \mathbb{N}. a(\bar{d}^k(\bar{x}))] \quad - f \text{ terminates on } \bar{x}$$

always-terminating "version" of  $f$  (a logic-mode function, if  $a, b, c, d$  are logic-mode):

$$\hat{f}(\bar{x}) \triangleq \text{if } t(\bar{x}) \text{ then } [\text{if } a(\bar{x}) \text{ then } b(\bar{x}) \text{ else } c(\bar{x}, \hat{f}(\bar{d}(\bar{x})))] \text{ else } \dots \quad \text{any value (irrelevant)}$$

$$\mu_f^{\wedge}(\bar{x}) \triangleq \text{if } t(\bar{x}) \text{ then } \min \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \text{ else } \dots \quad \text{any natural number (irrelevant)} \quad \mu_f^{\wedge}: U^n \rightarrow \mathbb{N}$$

$$\prec_f^{\wedge} \triangleq \prec \subseteq \mathbb{N} \times \mathbb{N}$$

$$\vdash \boxed{\mu\text{-end}} \quad t(\bar{x}) \Rightarrow a(\bar{d}^{\mu_f^{\wedge}(\bar{x})}(\bar{x}))$$

$$\vdash t(\bar{x}) \xrightarrow{\delta_{\mu_f^{\wedge}}} \mu_f^{\wedge}(\bar{x}) = \min \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow \mu_f^{\wedge}(\bar{x}) \in \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow a(\bar{d}^{\mu_f^{\wedge}(\bar{x})}(\bar{x}))$$

QED

$$\vdash \boxed{\mu\text{-min}} \quad a(\bar{d}^l(\bar{x})) \Rightarrow l \geq \mu_f^{\wedge}(\bar{x})$$

$$a(\bar{d}^l(\bar{x})) \xrightarrow{\delta_t} t(\bar{x}) \xrightarrow{\delta_{\mu_f^{\wedge}}} \mu_f^{\wedge}(\bar{x}) = \min \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\}$$

$$\hookrightarrow l \in \{k \in \mathbb{N} \mid a(\bar{d}^k(\bar{x}))\} \longrightarrow l \geq \mu_f^{\wedge}(\bar{x})$$

QED

$$\vdash \boxed{\tau_f^{\wedge}} \quad t(\bar{x}) \wedge \neg a(\bar{x}) \Rightarrow \mu_f^{\wedge}(\bar{d}(\bar{x})) \prec_f^{\wedge} \mu_f^{\wedge}(\bar{x})$$

$$t(\bar{x}) \xrightarrow{\mu\text{-end}} a(\bar{d}^{\mu_f^{\wedge}(\bar{x})}(\bar{x})) \longrightarrow a(\bar{d}^{\mu_f^{\wedge}(\bar{x})-1}(\bar{d}(\bar{x}))) \xrightarrow[\mu\text{-min}]{} \mu_f^{\wedge}(\bar{x}) - 1 \geq \mu_f^{\wedge}(\bar{d}(\bar{x})) \longrightarrow \mu_f^{\wedge}(\bar{d}(\bar{x})) \geq \mu_f^{\wedge}(\bar{d}(\bar{x})) + 1 > \mu_f^{\wedge}(\bar{d}(\bar{x})) \xrightarrow{\delta_{\prec_f^{\wedge}}} \mu_f^{\wedge}(\bar{d}(\bar{x})) \prec_f^{\wedge} \mu_f^{\wedge}(\bar{x})$$

$$\neg a(\bar{x}) \longrightarrow \neg a(\bar{d}^0(\bar{x})) \xrightarrow{\mu_f^{\wedge}(\bar{x}) \neq 0}$$

QED

alternative recursive definition of the measure function:

$$\varphi(u) \triangleq \begin{cases} u & \text{if } u \in \mathbb{N} \\ 0 & \text{else} \end{cases} \quad - \text{fixing function for } \mathbb{N}$$

$$\varepsilon_t(\bar{x}) \triangleq \exists k. \alpha(\bar{d}^{\varphi(k)}(\bar{x})) \quad - \text{witness of } t, \text{ for slightly modified definition } t(\bar{x}) \triangleq [\exists k. \alpha(\bar{d}^{\varphi(k)}(\bar{x})]$$

$$\begin{aligned} v(\bar{x}, k) &\triangleq \underbrace{\text{let } \tilde{k} = \varphi(k) \text{ in } \text{if } \alpha(\bar{d}^{\tilde{k}}(\bar{x})) \vee \tilde{k} \geq \varphi(\varepsilon_t(\bar{x})) \text{ then } \tilde{k} \text{ else } v(\bar{x}, \tilde{k}+1)}_{\substack{\text{recursively find min } k \text{ such that } \alpha(\bar{d}^k(\bar{x})), \text{ if } t(\bar{x}) \\ \text{stop at } \varphi(\varepsilon_t(\bar{x})) \text{ anyhow; min } k \text{ is always found}}} \\ \mu_v(\bar{x}, k) &\triangleq \varphi(\varepsilon_t(\bar{x}) - \varphi(k)) \quad \prec_v \triangleq \subset \mathbb{N} \times \mathbb{N} \quad \left. \begin{array}{l} \text{if } t(\bar{x}), \text{ because } \min k \leq \varphi(\varepsilon_t(\bar{x})) \\ \text{if } t(\bar{x}), \text{ because } \min k \leq \varphi(\varepsilon_t(\bar{x})) \end{array} \right\} \end{aligned}$$

$$\begin{aligned} \vdash \boxed{v} \quad & \neg \alpha(\bar{d}^{\varphi(k)}(\bar{x})) \wedge \varphi(k) < \varphi(\varepsilon_t(\bar{x})) \Rightarrow \mu_v(\bar{x}, \varphi(k)+1) \prec_v \mu_v(\bar{x}, k) \\ \mu_v(\bar{x}, \varphi(k)+1) &\stackrel{\delta_{r_v}}{=} \varphi(\varepsilon_t(\bar{x}) - \varphi(\varphi(k)+1)) \stackrel{\delta_\varphi}{=} \varphi(\varepsilon_t(\bar{x}) - \varphi(k)-1) \stackrel{\delta_\varphi}{=} \varepsilon_t(\bar{x}) - \varphi(k)-1 \\ \varphi(k) < \varphi(\varepsilon_t(\bar{x})) &\Rightarrow \varphi(\varepsilon_t(\bar{x})) \neq 0 \stackrel{\delta_\varphi}{\rightarrow} \varepsilon_t(\bar{x}) \in \mathbb{N} \stackrel{\delta_\varphi}{\rightarrow} \varphi(k) < \varepsilon_t(\bar{x}) \rightarrow \varepsilon_t(\bar{x}) - \varphi(k) > 0 \\ \mu_v(\bar{x}, k) &\stackrel{\delta_{r_v}}{=} \varphi(\varepsilon_t(\bar{x}) - \varphi(k)) \stackrel{\delta_\varphi}{=} \varepsilon_t(\bar{x}) - \varphi(k) \rightarrow \mu_v(\bar{x}, \varphi(k)+1) \prec_v \mu_v(\bar{x}, k) \leftarrow \delta_{r_v} \end{aligned}$$

QED

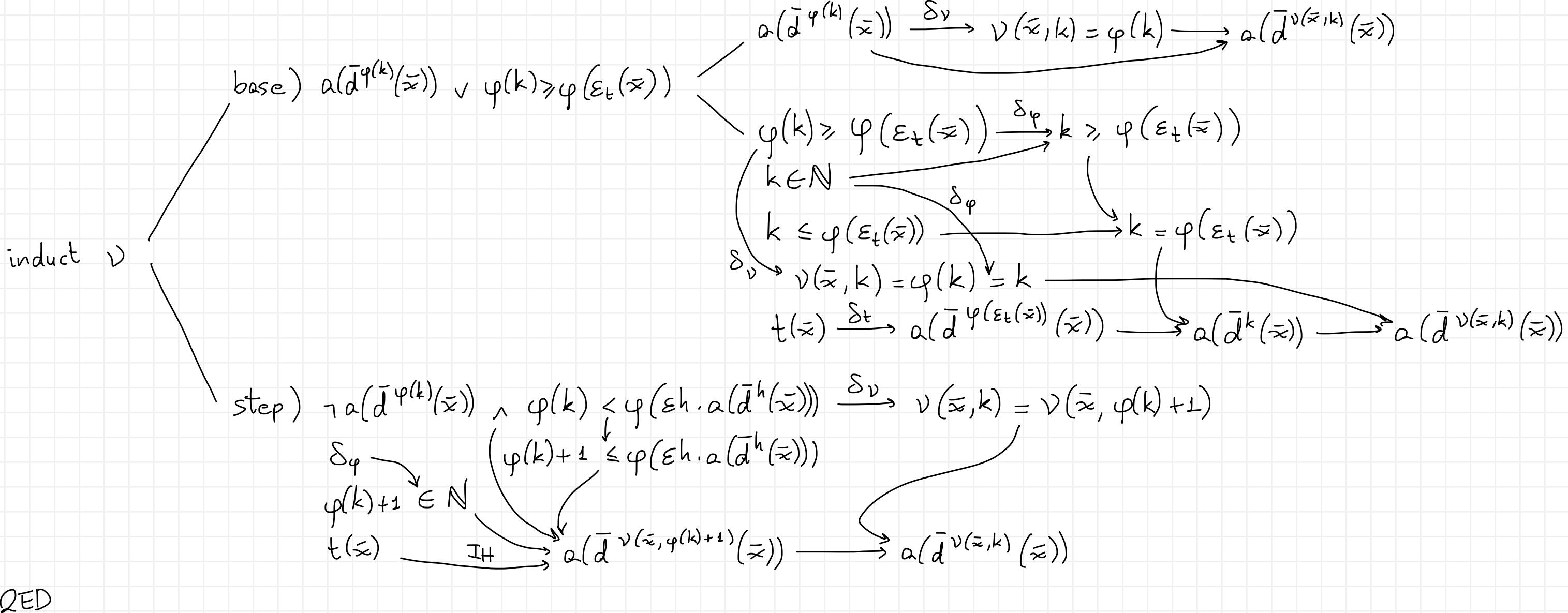
$$\begin{aligned} \vdash \boxed{v \in \mathbb{N}} \quad & v(\bar{x}, k) \in \mathbb{N} \\ \text{induct } v \quad & \begin{array}{c} \text{base) } \delta_v \rightarrow v(\bar{x}, k) = \varphi(k) \rightarrow v(\bar{x}, k) \in \mathbb{N} \\ \delta_\varphi \rightarrow \varphi(k) \in \mathbb{N} \end{array} \\ \text{step) } \quad & \delta_v \rightarrow v(\bar{x}, k) = v(\bar{x}, \varphi(k)+1) \rightarrow v(\bar{x}, k) \in \mathbb{N} \\ & \text{IH } \rightarrow v(\bar{x}, \varphi(k)+1) \in \mathbb{N} \end{array}$$

QED

$$\mu_f^*(\bar{x}) \triangleq v(\bar{x}, 0) \quad - \text{alternative definition of } \mu_f^*$$

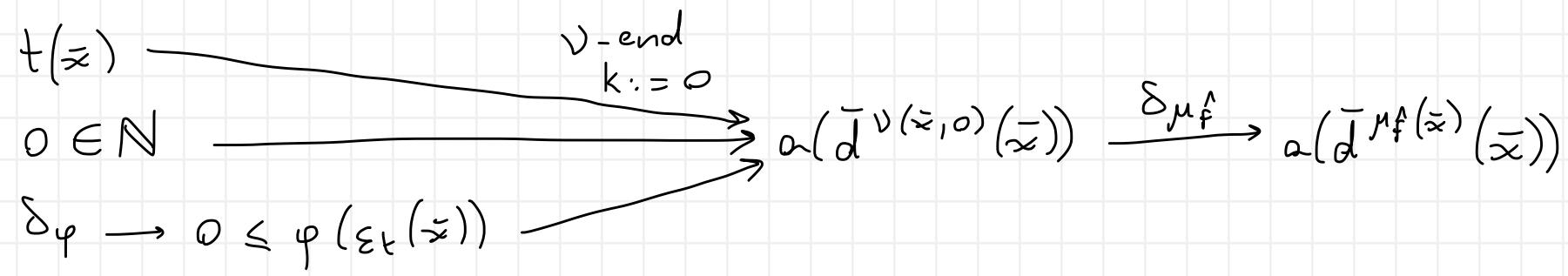
$$\mu_f^* : \mathcal{U}^n \rightarrow \mathbb{N} \quad (\Leftarrow v \in \mathbb{N})$$

$\vdash \boxed{\nu\text{-end}} \quad t(\bar{x}) \wedge k \in \mathbb{N} \wedge k \leq \varphi(\varepsilon_t(\bar{x})) \Rightarrow \alpha(\bar{d}^{\nu(\bar{x}, k)}(\bar{x}))$



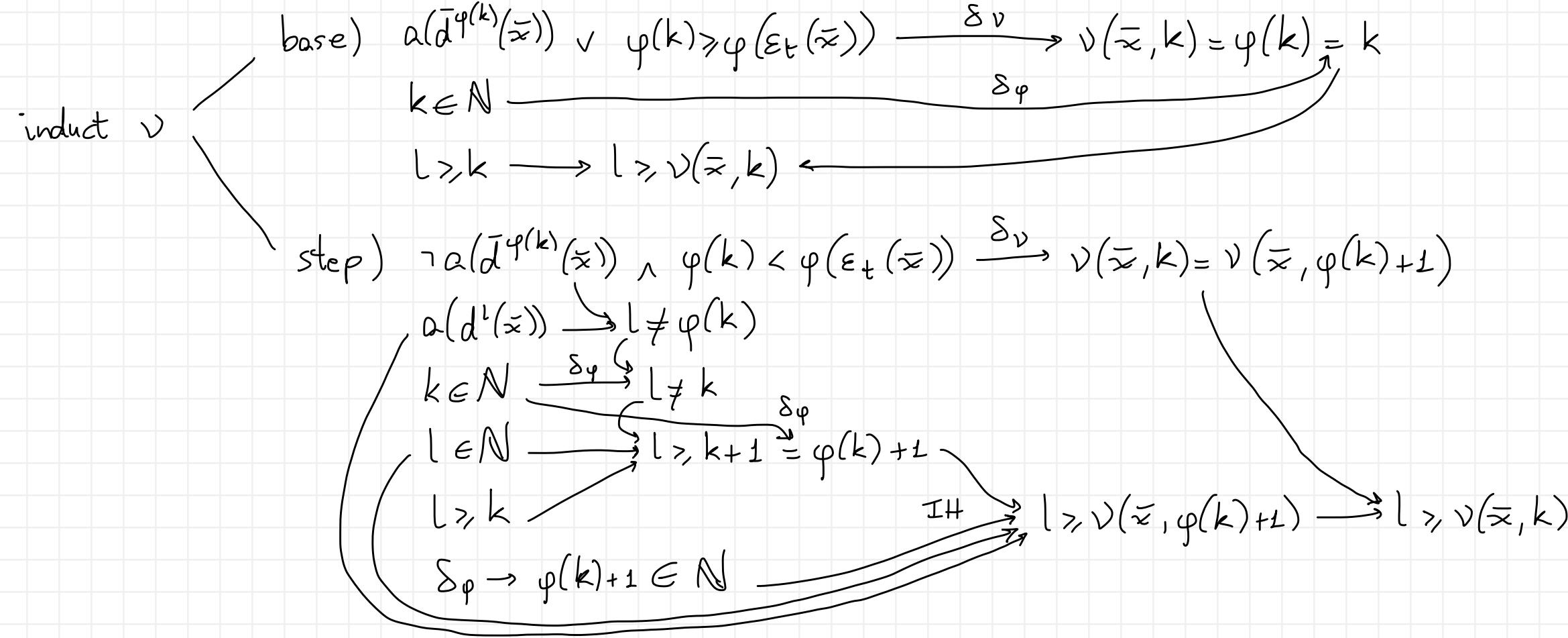
QED

$\vdash \boxed{\mu\text{-end}} \quad t(\bar{x}) \Rightarrow \alpha(\bar{d}^{\mu_f(\bar{x})}(\bar{x})) \quad - \quad \text{alternative proof for the alternative definition of } \mu_f^\wedge$



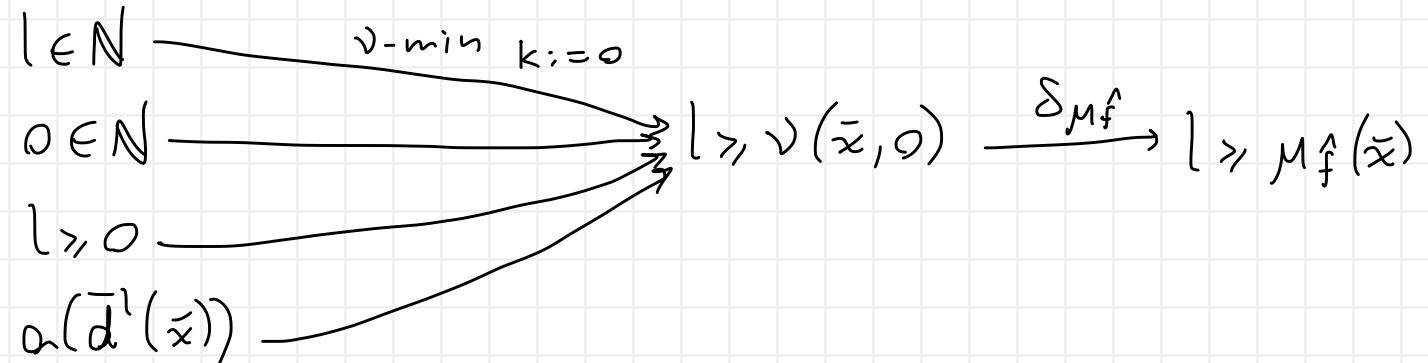
QED

$\vdash \boxed{\nu\text{-min}} \quad l \in \mathbb{N} \wedge k \in \mathbb{N} \wedge l > k \wedge a(d^l(\bar{x})) \Rightarrow l > \nu(\bar{x}, k)$



QED

$\vdash \boxed{\mu\text{-min}} \quad a(d^{-l}(\bar{x})) \Rightarrow l > \mu_f^\wedge(\bar{x}) \quad - \quad \text{alternative proof for the alternative definition of } \mu_f^\wedge$



QED

$\tau_f^\wedge$  proved as before using  $\mu\text{-end}$  and  $\mu\text{-min}$