

An Integration of Axiomatic Set Theory with ACL2

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UT Austin (retired)

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OUTLINE

Introduction

Axioms and Basic Notions

Review of First Talk

Embedding ACL2 in ZFG

Comprehension Scheme via Z_{sub}

Developing More Set Theory

Replacement Scheme via Z_{fn} , with Applications

$Z_{\text{if}y}$

Two Classical Examples

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General Information

Motivation

About Set Theory and ACL2

Examples

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Books use no [trust tags](#) and required **no ACL2 changes**.

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Key new insight last Fall: ACL2 can be a pure set-theory prover by encoding ACL2 primitives and data into set theory.

Additional motivation: Provides a vehicle for embedding higher-order logic (HOL) developments into ACL2.

- ▶ That could be the subject of future talks.

ABOUT SET THEORY AND ACL2

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 $n = \{0, 1, \dots, n - 1\}$.
- ▶ Other ACL2 objects are encoded as discussed later, e.g.:
 - ▶ Cons is represented using the Kuratowski ordered pair:
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Let's look at this picture from Wikipedia:

$V_{\omega * \omega}$

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- ▶ Classical set theory example: **Cantor's theorem** (Let's look briefly at the certifiable book, `cantor.lisp`); we'll revisit it later after providing more background.
- ▶ "Higher-order function" example: `map`
 - ▶ We'll look at `(defun map ...)` in `base.lisp` and the two theorems following it. First note:
 - ▶ In `(map f lst)`, think of `f` as a set of ordered pairs and `lst` as an ACL2 list.
 - ▶ I'll explain later how `(defthmz ... :props ...)` can be viewed as `(defthm ...)`.
 - ▶ We'll look at `zify.lisp` to see an application of `map` to the Fibonacci function.
 - ▶ Later we may look at `foldr.lisp`.

OUTLINE

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Z_{ify}

Two Classical Examples

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OUTLINE

Axioms and Basic Notions
ZFG

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Goal: Provide a platform for efficient set-theory reasoning.

- ▶ The axioms need justification, but need not be minimal.
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Let's look at the exports in the first encapsulate form in `base.lisp`, up to "Embedding of ACL2 data types".

- ▶ Notice the local witness of `nil` for `zfc`, which serves as a hypothesis!
- ▶ A metatheoretic argument provides a meaningful interpretation for which `(zfc)` is true.
- ▶ Not included there: Comprehension (Subset) or Replacement (equivalently, Collection) schemes of ZF (to be discussed later)

OUTLINE

Introduction

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Review of First Talk

Embedding ACL2 in ZFG

Comprehension Scheme via Z_{sub}

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Replacement Scheme via Z_{fn} , with Applications

Z_{ify}

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We'll look again at the initial `encapsulate` event in `base.lisp`.

It introduces *hypothesis function* `zfc` and primitives `in`, `pair`, `min-in`, `union`, `omega`, and `powerset`, along with `subset` and some basic axioms.

OUTLINE

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Developing More Set Theory

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Z_{ify}

Two Classical Examples

Future Work and Wrapping Up

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Logical Overview

Encoding ACL2 Objects in Set Theory

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I'll refer to these foundations — where ACL2 objects are encoded as sets and ACL2 functions are defined in ZFG — as *our underlying set theory*.

ENCODING ACL2 OBJECTS IN SET THEORY

Let's look at the rest of that initial `encapsulate` in `base.lisp` to see how ACL2 data type recognizers are defined — and also at the definitions of `relation-p`, `funp`, and `apply` after that.

ENCODING ACL2 OBJECTS IN SET THEORY

Let's look at the rest of that initial `encapsulate` in `base.lisp` to see how ACL2 data type recognizers are defined — and also at the definitions of `relation-p`, `funp`, and `apply` after that.

TIP: Note, as in `funp`, the use of `non-exec` in `defun-sk` to support `guard` verification.

OUTLINE

Introduction

Axioms and Basic Notions

Review of First Talk

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Developing More Set Theory

Replacement Scheme via Z_{fn} , with Applications

Z_{ify}

Two Classical Examples

Future Work and Wrapping Up

OUTLINE

Comprehension Scheme via Z_{sub}

Comprehension in ZF

Z_{sub} Example

More Z_{sub} Examples

$Z_{\text{fc-table}}$

Defthmz and :Props

Defthmz Examples

Simplifying Exports from Z_{sub}

COMPREHENSION IN ZF

The *Comprehension* (or *Subset*) scheme of ZF says that the intersection of a predicate with a set is a set.

- ▶ Informally: $\{a \in x : P(a)\}$ is a set.
- ▶ Formal statement, for each formula P with y not free:
$$\forall x \exists y \forall a (a \in y \Leftrightarrow (a \in x \wedge P))$$
- ▶ I'll call x the *bounding set*.

ZSUB EXAMPLE (1)

From base.lisp:

```
; The following defines the Cartesian product  
; (prod2 a b)  
; as:  
; {p \in (powerset (powerset (union2 a b))) :  
; (prod-member p a b)}
```

```
(zsub prod2 (a b)  
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  (prod-member p a b)  
)
```

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For Comprehension (and `zsub`), we always need a *bounding set*. Why does `(powerset (powerset (union2 a b)))` serve that purpose?

ZSUB EXAMPLE (2)

Again, why is $a \times b$ contained in
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So $\{\{x\}, \{x, y\}\}$ is a subset of (powerset (union2 a b)),
hence it's in (powerset (powerset (union2 a b))).

ZSUB EXAMPLE (3)

```
(zsub prod2 (a b)
  p
  (powerset (powerset (union2 a b)))
  (prod-member p a b)
)
```

Let's see how this call of `zsub` expands, using `:trans1` and focusing on `PROD2$COMPREHENSION`.

MORE Z_{SUB} EXAMPLES

As time permits we'll take a quick look at more examples in
`base.lisp`:

`domain, inverse, image, compose`

ZFC-TABLE

Recall that the `prod2` example above generates:

```
(TABLE ZFC-TABLE
  ' PROD2$PROP
  ' (ZSUB PROD2 (A B)
    P
    (POWERSET (POWERSET (UNION2 A B)))
    (PROD-MEMBER P A B)))
```

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Thus: The `table` guard of `zfc-table` checks that `prod2$prop` can be assumed to hold by the Comprehension scheme.

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 - ▶ After all, adding a bunch of \top hypotheses has no logical effect.
- ▶ The default value for `:props` is `(zfc)`.
- ▶ `Defthmdz` and `thmz` similarly extend `defthmd` and `thm` (respectively) with a `:props` argument.

DEPTHMZ EXAMPLES

Use `:trans1` to look at examples in `base.lisp`, e.g.,
`ordinal-p-omega` and `in-prod2`.

DEFTHMZ EXAMPLES

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Make-event **tips** from

```
:trans1 (CHECK-PROPS DEFTHMZ (ZFC PROD2$PROP)):
```

- ▶ **TIP:** Use `:expansion?` to avoid bloat in `.cert` file.
- ▶ **TIP:** Use `:on-behalf-of :quiet` to suppress noisy output
- ▶ **TIP:** Use `:check-expansion t` to ensure that the check is made even at `include-book` time.

SIMPLIFYING EXPORTS FROM Z_{SUB}

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Evaluate `:pe prod2$comprehension` and compare to `in-prod2`.

SIMPLIFYING EXPORTS FROM ZSUB

Evaluate `:pe prod2$comprehension` and compare to `in-prod2`.

Let's look at the proof of `in-prod2` (file `base.lisp`), which simplifies `prod2$comprehension`.

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Z_{ify}

Two Classical Examples

Future Work and Wrapping Up

OUTLINE

Developing More Set Theory

Ordinals

Iterated Composition

De Morgan's Laws Etc.

Function Spaces

Reasoning about Free Variables and Quantifiers

Transfinite Induction

ORDINALS

Let's look at these key events in the section "Omega is an ordinal" in `base.lisp` (with "`:guard t`" omitted).

```
(defun-sk in-is-linear (s)
  (forall (x y) (implies (and (in x s)
                               (in y s)
                               (not (equal x y)))
                        (or (in x y)
                            (in y x)))))

(defun-sk transitive (x)
  (forall a (implies (in a x)
                    (subset a x)))

  :rewrite :direct)

(defun ordinal-p (x)
  (and (in-is-linear x)
       (transitive x)))

(defthmz ordinal-p-omega (ordinal-p (omega)))
```

ORDINALS (CONTINUED)

See `ordinals.lisp` for a few more theorems about ordinals.

A key result:

```
(defthmz ordinal-trichotomy
  (implies (and (ordinal-p a)
                (ordinal-p b)
                (not (in a b))
                (not (in b a)))
           (equal (equal a b)
                  t))
  :props (zfc diff$prop)
  :hints ...)
```

ITERATED COMPOSITION

See `iterate.lisp`.

DE MORGAN'S LAWS ETC.

See `set-algebra.lisp`, e.g., De Morgan's Laws.

FUNCTION SPACES

See `fun-space.lisp`.

REASONING ABOUT FREE VARIABLES AND QUANTIFIERS

Example: see `demo1.lisp` for a proof of the following theorem from `set-algebra.lisp`.

```
(defthmz domain-union2
  (equal (domain (union2 r s))
         (union2 (domain r) (domain s))))
:props (zfc domain$prop)
:hints ...)
```

FREE VARIABLES AND QUANTIFIERS (CONTINUED)

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- ▶ **TIP:** Enable `extensionality-rewrite` to prove two sets are equal.
- ▶ **TIP:** Use `:otf-flg t` to see all checkpoints.
- ▶ **TIP:** Use `skip-proofs` to formulate lemmas (maybe with numbering scheme X-i-j-k...) and then see if those suffice.

FREE VARIABLES AND QUANTIFIERS (CONTINUED)

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TRANSFINITE INDUCTION

Time permitting, we'll talk about *epsilon-induction* and look at the macro `prove-inductive-suffices` and the examples below it in `induction.lisp`.

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Transfinite induction on the ordinals is a special case of epsilon-induction.

OUTLINE

Introduction

Axioms and Basic Notions

Review of First Talk

Embedding ACL2 in ZFG

Comprehension Scheme via Z_{sub}

Developing More Set Theory

Replacement Scheme via Z_{fn} , with Applications

Z_{ify}

Two Classical Examples

Future Work and Wrapping Up

OUTLINE

Replacement Scheme via Zfn , with Applications

Replacement Application #1: The Cumulative Hierarchy

Replacement in ZF and in ACL2

All Good ACL2 Objects Are in V_ω

The Good ACL2 Objects Form a Set

Replacement Example #2: Transitive Closure

REPLACEMENT APPLICATION #1:

THE CUMULATIVE HIERARCHY

The book `base.lisp` defines $V = \{\langle x, y \rangle : x \in \omega \wedge y = V_{map}(x)\}$
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(defun v-map (n) ; uses ordinary ACL2 recursion!  
  (declare (type (integer 0 *) n))  
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      0  
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Let's take a look using `:trans1`:

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    t))
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         t))  
  
(defun v-omega () ; union of v-map(0), v-map(1), ...  
  (declare (xargs :guard t))  
  (union (image (v))))
```

REPLACEMENT IN ZF AND IN ACL2

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Here's what Wikipedia says, but we'll look at [the picture](#).

$$\forall w_1, \dots, w_n \forall A ([\forall x \in A \exists! y \phi(x, y, w_1, \dots, w_n, A)] \implies \\ \exists B \forall y [y \in B \Leftrightarrow \exists x \in A \phi(x, y, w_1, \dots, w_n, A)])$$

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ACL2 modifies the Replacement Scheme as follows. ([This ACL2 version follows easily from the ZF axioms.](#))

- ▶ F can associate more than one value with the same input;
- ▶ the mapping can be **undefined** on any elements of A ; and
- ▶ the result is a **set-theoretic function** — a set of ordered pairs — based on the restriction of F to A .

ALL GOOD ACL2 OBJECTS ARE IN V_ω

Recognizer for “good ACL2 object”:

```
(defun acl2p (x)
  (declare (xargs :guard t))
  (cond ((consp x) (and (acl2p (car x))
                        (acl2p (cdr x))))
        (t (not (bad-atom x)))))
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```

Good ACL2 objects sit in V_ω (see `base.lisp`), which is critical for the use of `zsub` on the next slide:

```
(defthmz v-omega-contains-acl2p
  (implies (acl2p x)
            (in x (v-omega))))
:props ... :hints ...)
```

THE GOOD ACL2 OBJECTS FORM A SET

We'll use the following notion later (see `mirror` example).

`(acl2)` = $\{x \in V_\omega : \text{acl2p}(x)\}$ — see `zify.lisp`.

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(zsub acl2 ()      ; name, args
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  (acl2p x) ) ; the property
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```

```
  ; strengthens acl2$comprehension
```

```
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```

```
         (acl2p x))
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```
  :props (zify-prop acl2$prop v$prop))
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```

```
    (acl2p x))
```

```
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```

```
(in-theory (disable acl2$comprehension))
```

REPLACEMENT EXAMPLE #2: TRANSITIVE CLOSURE

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A set is *transitive* if every member of a member is a member, i.e., every member is a subset.

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Time permitting, we'll look at theorems labeled "A key theorem" in `tc.lisp`.

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Time permitting, we'll look at theorems labeled "A key theorem" in `tc.lisp`.

Perhaps we'll also look at the definition of `tc` in file `tc.lisp`.

```
(defun tc-n (n s) ...)
(zfn tc-fn (s) ...)
(defun tc (s) ...)
```


OUTLINE

Introduction

Axioms and Basic Notions

Review of First Talk

Embedding ACL2 in ZFG

Comprehension Scheme via Z_{sub}

Developing More Set Theory

Replacement Scheme via Z_{fn} , with Applications

Z_{if}

Two Classical Examples

Future Work and Wrapping Up

OUTLINE

Zify

Zify Introduction: Revisiting fib

Zify Example: Mirror

Zify*

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Look at the `fib` example in `zify.lisp`.

```
:trans1 (zify zfib fib :dom (omega) :ran (omega))
```

Below is a key part of the `zify` call above, informally:

$$(zfib) = \{\langle p_1, p_2 \rangle \in \omega \times \omega : p_2 = fib(p_1)\}.$$

```
(zsub zfib ()  
  p  
  (prod2 (omega) (omega))  
  (equal (cdr p) (fib (car p))))
```

ZIFY EXAMPLE: MIRROR (1)

For `zify`, the defaults for `:dom` and `:ran` — the domain and image (range) — are `(acl2)`, the set of good ACL2 objects.

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  (cond ((atom x) x)
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(prove-acl2p mirror) ; (ACL2P X)  $\implies$  (ACL2P (MIRROR X))
(zify zmirror mirror)
```


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- ▶ `:trans*` `t` `(prove-acl2p mirror)`
- ▶ **TIP:** Use `:trans*` instead of `:trans1` when `make-event` is involved in the expansion.

ZIFY EXAMPLE: MIRROR (2)

Now we can *attempt to prove*:

```
(thm (equal (apply (zmirror) '(a . b) . (c . d)))  
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```

ZIFY EXAMPLE: MIRROR (2)

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(thm (equal (apply (zmirror) '((a . b) . (c . d)))  
            '((d . c) . (b . a))))
```

Fix it using `thmz` (and show `zify-prop`):

```
(thmz (equal (apply (zmirror) '((a . b) . (c . d)))  
            '((d . c) . (b . a)))  
      :props (zify-prop acl2$prop v$prop zmirror$prop))
```

Z I F Y *

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`Zify*` is a variant of `zify` that can convert arbitrary-arity
ACL2 functions to set-theoretic functions.

- ▶ The idea is to get a unary function that maps arglists to values.

You can see `zify.lisp` for a few examples, but time
permitting we'll look at `foldr` in `foldr.lisp`.

ZIFY* (CONTINUED)

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Let's take a quick look at the book, `foldr.lisp`, up through:

```
(thmz
  (implies (acl2p lst)
            (equal (foldr lst (zbinary-*) 1)
                   (timeslist lst)))
  :props (foldr-prop zbinary-*$prop acl2$prop))
```

```
(thmz
  (implies (acl2p lst)
            (equal (foldr '(2 3 5) (zbinary-*) 1)
                   30))
  :props (foldr-prop zbinary-*$prop acl2$prop))
```

OUTLINE

Introduction

Axioms and Basic Notions

Review of First Talk

Embedding ACL2 in ZFG

Comprehension Scheme via Z_{sub}

Developing More Set Theory

Replacement Scheme via Z_{fn} , with Applications

Z_{ify}

Two Classical Examples

Future Work and Wrapping Up

OUTLINE

Two Classical Examples

Cantor's Theorem

The Schröder-Bernstein Theorem

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- ▶ Note the natural use of `zsub` to follow the Wikipedia proof.
- ▶ **TIP:** Note the use of `minimal-theory` for control of the proof.
- ▶ **TIP:** It's OK to leave `proof-builder` `:instructions` when they're easily maintainable.

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- ▶ **TIP:** Locally included book `*-support.lisp` has ugly details (a technique used earlier in the `rtl` books and elsewhere).
- ▶ **TIP:** Hand proofs can be helpful; see `schroeder-bernstein-main-2-2` in `schroeder-bernstein-support.lisp`.

OUTLINE

Introduction

Axioms and Basic Notions

Review of First Talk

Embedding ACL2 in ZFG

Comprehension Scheme via Z_{sub}

Developing More Set Theory

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OUTLINE

Future Work and Wrapping Up
Future Work (Highly Incomplete List!)
Wrapping Up

FUTURE WORK (HIGHLY INCOMPLETE LIST!)

- ▶ Transfinite recursion, e.g., V_α for all ordinals α
- ▶ Cardinals, cardinality (in progress)
- ▶ Higher-order applications (e.g., temporal logics)
- ▶ Tool improvements, e.g., let `zify` return `:REDUNDANT`
- ▶ More automation
 - ▶ ACL2 modification for parity-based rewriting (or maybe use existing clause-processor?)
 - ▶ Quantifier instantiation (maybe Dave Greve's stuff?)
 - ▶ Automated functional instantiation (maybe Joosten et al.'s 2013 workshop paper on `instance-of-defspec`)
- ▶ Prove correctness for the embedding of ACL2 into ZFG.
- ▶ More set theory
 - ▶ ω_1 (soon; should be easy using Cantor's theorem)
 - ▶ Cofinality, closed unbounded subsets, stationary sets
 - ▶ Mostowski collapse
 - ▶ Independence results
 - ▶ ...
- ▶ Other math, e.g., basic topology

WRAPPING UP

Possible PhD dissertation topic(s)?

Collaborators?

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Thank you for your attention!