A Mechanical Proof of the Chinese Remainder Theorem

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Informal Statement

Theorem *Let $m_1, \ldots, m_k \in \mathbb{N}$ be pairwise relatively prime moduli and let $a_1, \ldots, a_k \in \mathbb{N}$. There exists $x \in \mathbb{N}$ such that*

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_k \pmod{m_k}. \end{aligned}$$

If x' satisfies the same congruences, then

$$x' \equiv x \pmod{m_1 m_2 \cdots m_k}.$$

ACL2 Formalization

```
(defun g-c-d (x y)
  (declare (xargs :measure (nfix (+ x y))))
  (if (zp x)
      y
    (if (zp y)
        x
      (if (<= x y)
          (g-c-d x (- y x))
        (g-c-d (- x y) y)))))

(defun rel-prime (x y)
  (= (g-c-d x y) 1))

(defun congruent (x y m)
  (= (rem x m) (rem y m)))

(defun congruent-all (x a m)
  (if (endp m)
      t
    (and (congruent x (car a) (car m))
         (congruent-all x (cdr a) (cdr m)))))

(defthm chinese-remainder-theorem
    (implies (and (natp-all a)
                  (rel-prime-moduli m)
                  (= (len a) (len m)))
             (and (natp (crt-witness a m))
                  (congruent-all (crt-witness a m) a m))))
```

Informal Proof

Lemma 1 *If $x, y \in \mathbb{N}$ are relatively prime, then there exists $s \in \mathbb{Z}$ such that $sy \equiv 1 \pmod{x}$.*

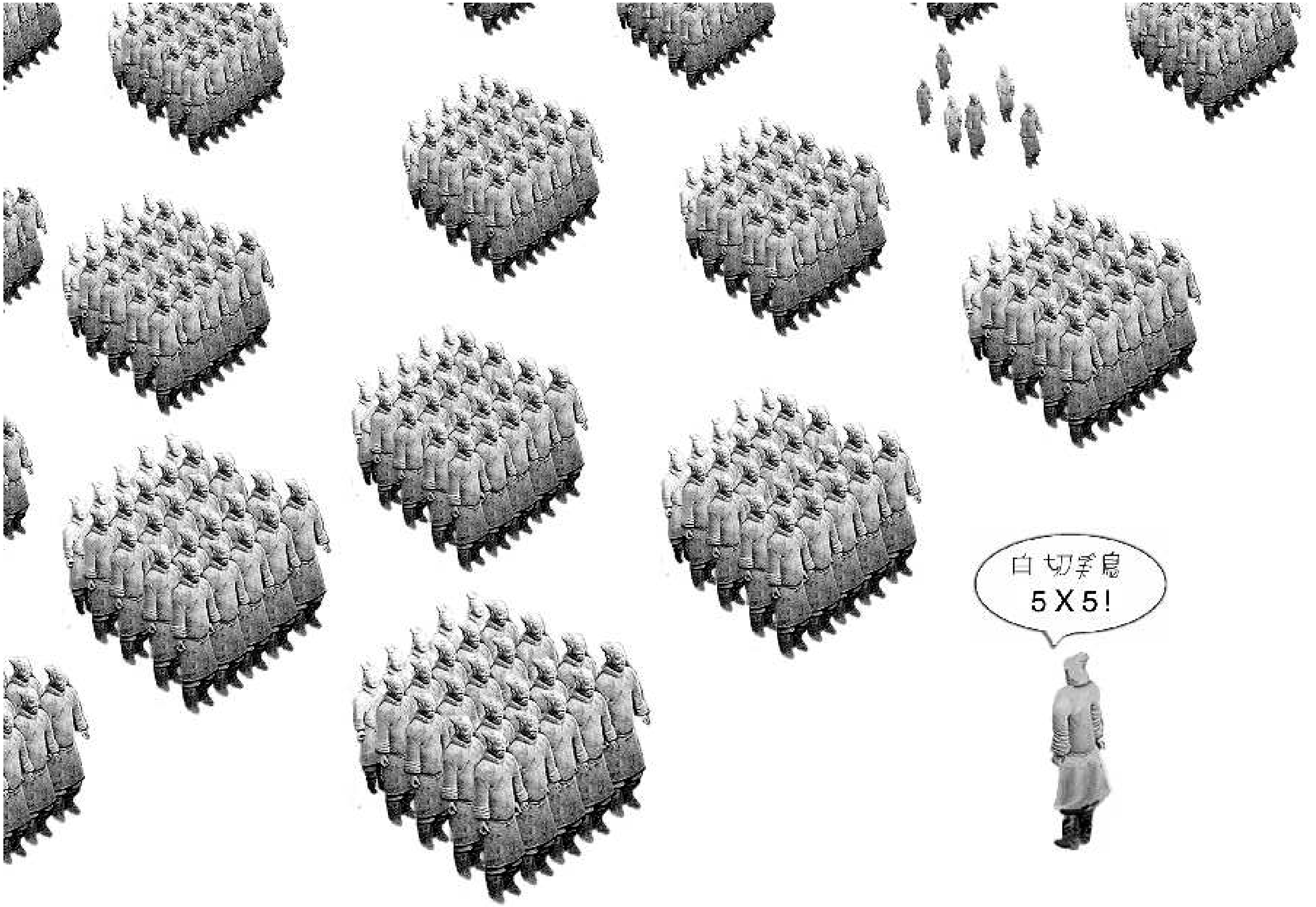
Lemma 2 *If $x, y, z \in \mathbb{N}$ and x is relatively prime to both y and z, then x is relatively prime to yz.*

Proof of CRT: Let $M = m_1 m_2 \cdots m_k$. For $i = 1, \ldots, k$, let $M_i = M/m_i$ and find s_i such that $s_i M_i \equiv 1 \pmod{m_i}$. Let

$$x = a_1 s_1 M_1 + a_2 s_2 M_2 + \cdots + a_k s_k M_k.$$

Then

$$x \equiv a_i s_i M_i \equiv a_i \pmod{m_i}.$$



$N \equiv 6 \pmod{25}$

Example

Suppose we have $10000 \leq N \leq 50000$ and

$$\begin{aligned} N &\equiv 6 \pmod{25} \\ N &\equiv 13 \pmod{36} \\ N &\equiv 28 \pmod{49} \end{aligned}$$

Then we may solve for N as follows:

$$\begin{aligned} M &= 25 \cdot 36 \cdot 49 = 44100 \\ M_1 &= 36 \cdot 49 = 1764 \\ M_2 &= 25 \cdot 49 = 1225 \\ M_3 &= 25 \cdot 36 = 900 \end{aligned}$$

$$\begin{aligned} 1764 s_1 \equiv 1 \pmod{25} &\Leftrightarrow 14 s_1 \equiv 1 \pmod{25} \Leftrightarrow s_1 \equiv 9 \pmod{25} \\ 1225 s_2 \equiv 1 \pmod{36} &\Leftrightarrow s_2 \equiv 1 \pmod{36} \\ 900 s_3 \equiv 1 \pmod{49} &\Leftrightarrow 18 s_3 \equiv 1 \pmod{49} \Leftrightarrow s_3 \equiv 30 \pmod{49} \end{aligned}$$

$$a_1 = 6,\ a_2 = 13,\ a_3 = 28$$

$$\begin{aligned} x &= a_1 M_1 s_1 + a_2 M_2 s_2 + a_3 M_3 s_3 \\ &= 6 \cdot 1764 \cdot 9 + 13 \cdot 1225 \cdot 1 + 28 \cdot 900 \cdot 30 \\ &= 867281 \\ &\equiv 29281 \pmod{44100} \end{aligned}$$

$$N = 29281$$

Proof of Lemma 1

Lemma 1 *If $x, y \in \mathbb{N}$ are relatively prime, then there exists $s \in \mathbb{Z}$ such that $sy \equiv 1 \pmod{x}$.*

This is a special case of the following:

> *For all $x, y \in \mathbb{N}$, there exist $r, s \in \mathbb{Z}$ such that $rx + sy = gcd(x, y)$.*

The proof is by induction on $x + y$:

(1) If $x = 0$, then $r = 0$ and $s = 1$.

(2) If $y = 0$, then $r = 1$ and $s = 0$.

(3) If $0 < x \leq y$, then find r' and s' such that

$$r'x + s'(y - x) = gcd(x, y - x) = gcd(x, y)$$

and let $r = r' - s'$ and $s = s'$. Then

$$rx + sy = (r' - s')x + s'y = r'x + s'(y - x) = gcd(x, y).$$

(4) If $0 < y < x$, then find r' and s' such that

$$r'(x - y) + s'y = gcd(x - y, y) = gcd(x, y)$$

and let $r = r'$ and $s = s' - r'$.

Formal Proof

```
(mutual-recursion
 (defun r (x y)
   (declare (xargs :measure (nfix (+ x y))))
   (if (zp x)
       0
     (if (zp y)
         1
       (if (<= x y)
           (- (r x (- y x)) (s x (- y x)))
         (r (- x y) y)))))

 (defun s (x y)
   (declare (xargs :measure (nfix (+ x y))))
   (if (zp x)
       1
     (if (zp y)
         0
       (if (<= x y)
           (s x (- y x))
         (- (s (- x y) y) (r (- x y) y))))))
)

(defthm r-s-lemma
    (implies (and (natp x)
                  (natp y))
             (= (+ (* (r x y) x)
                   (* (s x y) y))
                (g-c-d x y))))
```

Proof of Lemma 2

Lemma 2 *If $x, y, z \in \mathbb{N}$ and x is relatively prime to both y and z, then x is relatively prime to yz.*

This is a consequence of the following basic properties of *gcd* and primes:

(1) *$gcd(x, y)$ divides both x and y.*

(2) *If d divides both x and y, then d divides $gcd(x, y)$.*

(3) *If $x > 1$, then some prime divides x.*

(4) *If a prime p divides ab, then p divides either a or b.*

It would take some work to prove these in ACL2. Fortunately, there is a more direct route to CRT.

Alternate Approach

Lemma 3 *Let $x, y_1, y_2, \ldots, y_k \in \mathbb{N}$ and $p = y_1 \cdots y_k$. If x is relatively prime to each y_i, then there exist $c, d \in \mathbb{Z}$ such that $cx + dp = 1$.*

Proof: Let $p' = y_1 \cdots y_{k-1}$. Assume that

$$rx + sy_k = 1$$

and, by induction, that

$$c'x + d'p' = 1.$$

Then

$$\begin{aligned}(sd')p &= (sy_k)(d'p') \\ &= (1 - rx)(1 - c'x) \\ &= 1 - (r + c' - rc'x)x.\end{aligned}$$

Thus, if $c = r + c' - rc'x$ and $d = sd'$, then

$$cx + dp = 1.$$

Formal Proof

```
(defun c (x l)
  (if (endp l)
      0
    (- (+ (r x (car l))
          (c x (cdr l)))
       (* (r x (car l))
          (c x (cdr l))
          x))))

(defun d (x l)
  (if (endp l)
      1
    (* (s x (car l))
       (d x (cdr l)))))

(defthm c-d-lemma
    (implies (and (natp x)
                  (natp-all l)
                  (rel-prime-all x l))
             (= (+ (* (c x l) x)
                   (* (d x l) (prod l)))
                1)))
```

Definition of `crt-witness`

```
(defun one-mod (x l)
  (* (d x l)
     (prod l)
     (d x l)
     (prod l)))

(defthm rem-one-mod-1
    (implies (and (natp x)
                  (> x 1)
                  (natp-all l)
                  (rel-prime-all x l))
             (= (rem (one-mod x l) x) 1)))

(defthm rem-one-mod-0
    (implies (and (natp x)
                  (> x 1)
                  (rel-prime-moduli l)
                  (rel-prime-all x l)
                  (member y l))
             (= (rem (one-mod x l) y) 0)))

(defun crt1 (a m l)
  (if (endp a)
      0
    (+ (* (car a) (one-mod (car m) (remove (car m) l)))
       (crt1 (cdr a) (cdr m) l))))

(defun crt-witness (a m) (crt1 a m m))
```

The Main Lemma

We prove the following generalization of CRT:

```
(defthm crt1-lemma
    (implies (and (natp-all a)
                  (rel-prime-moduli l)
                  (sublistp m l)
                  (= (len a) (len m)))
             (congruent-all (crt1 a m l) a m)))
```

The proof is by induction, as suggested by the definition:

```
(defun crt1 (a m l)
  (if (endp a)
      0
    (+ (* (car a) (one-mod (car m) (remove (car m) l)))
       (crt1 (cdr a) (cdr m) l))))
```

In the inductive case, the conclusion of the lemma expands as follows:

```
(and (congruent (+ (* (car a)
                      (one-mod (car m) (remove (car m) l)))
                   (crt1 (cdr a) (cdr m) l))
                (car a)
                (car m))
     (congruent-all (+ (* (car a)
                          (one-mod (car m) (remove (car m) l)))
                          (crt1 (cdr a) (cdr m) l))
                    (cdr a)
                    (cdr m))).
```

The Final Result

CRT is derived as an instance of `crt1-lemma`:

```
(defthm crt1-lemma
    (implies (and (natp-all a)
                  (rel-prime-moduli l)
                  (sublistp m l)
                  (= (len a) (len m)))
             (congruent-all (crt1 a m l) a m)))

(defthm chinese-remainder-theorem
    (implies (and (natp-all a)
                  (rel-prime-moduli m)
                  (= (len a) (len m)))
             (and (natp (crt-witness a m))
                  (congruent-all (crt a m) a m))))
```