Proof of Dickson's lemma in ACL2 via an explicit ordinal mapping

by Mátyás Sustik

I present the use of the ACL2 theorem prover to formalize and mechanically check a new proof of Dickson's lemma about monomial sequences. Dickson's lemma can be used to establish the termination of the Buchberger algorithm to find the Gröbner basis of a polynomial ideal. This effort is related to a larger project which aims to develop a mechanically verified computer algebra system.

Contents

Background

- A (polynomial) ideal I of an R ring is defined as a subset closed under subtraction and under multiplication with arbitrary elements of R. $i_1, i_2 \in I,\ r \in R \Longrightarrow i_1 - i_2 \in I,\ ir, ri \in I$.
- Classic example: modulo arithmetic in $\mathbb{Z}$. The ideal generated by 5 is the set: $\{\ldots, -10, -5, 0, 5, 10, \ldots\}$.
- Another example: the ideal generated by x^2 and $3x$ in $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients consists of the polynomials which have a constant coefficient equal to 0 and the coefficient of x is divisible by 3.

$$x^3 + 2x^2 + 6x = (x+2) \cdot x^2 + 2 \cdot 3x.$$

- The Gröbner basis is a uniquely determined special basis for a polynomial ideal; its determination helps to decide equality of ideals presented with arbitrary generators.
- Buchberger's algorithm takes an ideal given by a generator set and calculates the Gröbner basis.
- The termination of the algorithm is established by Dickson's lemma.
- Keith O. Geddes and S. R. Czapor and G. Labahn: Algorithms for Computer Algebra

Dickson's lemma

- We consider terms over a finite set of symbols e.g.: $x_0^2 x_1 x_2^3$.
- A monomial $x_0^{u_1} x_1^{u_2} \ldots x_{k-1}^{u_{k-1}}$ divides $x_0^{v_0} x_1^{v_1} \ldots x_{k-1}^{v_{k-1}}$ iff $u_i \leq v_i$ for all possible values of i.
- Claim: Given an infinite sequence of monomials: $m_1, m_2, m_3, \ldots$ exist i, j indices such that $i < j$ and m_i divides m_j.
- Classical proofs may use Ramsey's theorem about infinite graphs colored with finitely many colors, or other non-constructive arguments to select certain subsequences.

Dickson's lemma

(auxiliary material)

- Classical proof sketch 1: There is a subsequence $m_{i_1}, m_{i_2}, m_{i_3}, \ldots$ such that the exponent of the first variable in m_{i_j} is (weakly) increasing. Omit the first variable from the terms and restrict to the above subsequence to set the stage for an induction on the number of variables.
- Classical proof sketch 2: Suppose m_i does not divide m_j for any $i < j$. Denote the index of a 'witness' variable by $c(i, j)$: the exponent of the variable is less in m_j than in m_i. Consider the infinite complete graph on the positive integers naturally colored by $c(i, j)$. Ramsey's theorem asserts that there is an infinite uniformly colored complete subgraph, which would imply the existence of an infinite descending sequence of natural numbers, a contradiction.

Ordinals

- Panagiotis Manolios and Daron Vroon: Algorithms for Ordinal Arithmetic, 19th International Conference on Automated Deduction (CADE) 2003
- Additional ordinal lemmas. (About addition, exponentiation, and the notion of less than equal relation among ordinals.)

Ordinals

(auxiliary material)

- Ordinal addition is non-commutative, associative.
- $1 + \omega = \omega < \omega + 1$.
- Exponentiation is monoton.
- If $a_1 \leq a_2$ and $b_1 \leq b_2$ then $a_1 + b_1 \leq a_2 + b_2$.
- Suppose $a_1 < a_2$ and $b_1 \leq b_2$. Does this imply that $a_1 + b_1 < a_2 + b_2$?
- Suppose $a_1 \leq a_2$ and $b_1 < b_2$. Does this imply that $a_1 + b_1 < a_2 + b_2$?

Definition of the mapping

- Build an ordinal mapping which assigns an ordinal to the initial segments of the monomial sequence such that: if no monomial divides another appearing later in the sequence, then the ordinals form a decreasing sequence.

- I represent the monomials as k-tuples and denote by $\mathcal{A}_k$ the collection of finite sets of k-tuples:
$$\mathcal{A}_k = \{A \subset \mathbb{N}^k : |A| < \omega\}.$$

- We define the $M_k : \mathcal{A}_k \to Ord$ function inductively. If $A \in \mathcal{A}_1$ then set
$$M_1(A) = \min A,$$
with the agreement that $M_1(\{\}) = \omega$.

- Now suppose that $k > 1$ and that we have already defined M_{k-1}. For an arbitrary $A \in \mathcal{A}_k$ and $i \in \mathbb{N}$ define the $P_i^{(A)} \in \mathcal{A}_{k-1}$ sets and the $\alpha_i^{(A)}$ ordinals as follows:
$$P_i^{(A)} = \{(u_1, u_2, \ldots, u_{k-1}) : (u_0, u_1, \ldots, u_{k-1}) \in A, u_0 \leq i\},$$
$$\alpha_i^{(A)} = M_{k-1}(P_i^{(A)}).$$

Definition of the mapping

(auxiliary material)

- $M_1(\{3, 5, 2, 7\}) = 2$. Note that $\mathbb{N}^1 = \mathbb{N}$.

- $A = \{(3, 2), (2, 3), (1, 6), (2, 4), (3, 6)\}$.
$$\begin{array}{ll} P_0^{(A)} = \{\} & \alpha_0^{(A)} = \omega, \\ P_1^{(A)} = \{6\} & \alpha_1^{(A)} = 6, \\ P_2^{(A)} = \{3, 4, 6\} & \alpha_2^{(A)} = 3, \\ P_3^{(A)} = \{2, 3, 4, 6\} & \alpha_3^{(A)} = 2, \\ P_4^{(A)} = \{2, 3, 4, 6\} & \alpha_4^{(A)} = 2. \end{array}$$

Definition of the mapping

continued

- The P_i, α_i sequences stabilize for every $A \in \mathcal{A}_k$.
- Denote an index by $m = m^{(A)}$ for which $\alpha_i = \alpha_s$ for every $i \geq m$ and define:

$$M_k(A) = \left(\sum_{i=0}^{m-1} \omega^{\alpha_i}\right) + \omega^{\alpha_m+1}.$$

- Define the partial sums that make up M_k:

$$M_k(A, j) = \begin{cases} \min A & \text{if } k = 1 \\ \left(\sum_{i=j}^{m-1} \omega^{\alpha_i}\right) + \omega^{\alpha_m+1} & \text{if } k > 1 \end{cases}$$

Definition of the mapping

(auxiliary material)

- $A = \{(3,2),(2,3),(1,6),(2,4),(3,6)\}, m = 3.$
- $\alpha_0^{(A)} = \omega$, $\alpha_1^{(A)} = 6$, $\alpha_2^{(A)} = 3$, $\alpha_3^{(A)} = 2$.
- By the definition:

$$M_2(A) = \omega^\omega + \omega^6 + \omega^3 + \omega^3.$$

- The following form reveals the intuition behind the definition:

$$M_2(A) = \omega^\omega + \omega^6 + \omega^3 + \omega^2 + \omega^2 + \ldots.$$

Proof

1. If $A \subseteq B \in \mathcal{A}_k$ then $M_k(A) \geq M_k(B)$.
2. For any $A \in A_k$, $k > 1$ the α_i sequence is monotone decreasing.
3. For any $A, B \in \mathcal{A}_k$, $k > 1$, $M_k(A) = M_k(B)$ holds if and only if $\alpha_i^{(A)} = \alpha_i^{(B)}$ is true for every $i \in \mathbb{N}$.

   - Seven further lemmas lead to the proof of the above statement.
   - An induction scheme is specified.
   - Ordinal arithmetic lemmas are instantiated.

Proof

(continued)

4 If $A \subseteq B \in \mathcal{A}_k$, $v \in B \setminus A$ and $M_k(A) = M_k(B)$ then there is a $w \in A$ such that $w \leq_k v$.

- Witness functions are defined to allow formalization of some properties.
- If $x \in P_i^{(A)}$ then there exists a $y \in A$ such that:

$$y = (u_0, u_1, \ldots, u_{k-1}),\ \ u_0 \leq i,\ \ x = (u_1, \ldots, u_{k-1}).$$

5 If $s_1, s_2, \ldots, s_n$ is a finite sequence of k-tuples of natural numbers such that for any $1 \leq i < j \leq n$ we have $s_i \not\leq_k s_j$ then the $M_k(A_1), M_k(A_2), \ldots, M_k(A_n)$ sequence of ordinals is strictly decreasing where A_j denotes an initial segment of s_i: $A_j = \{s_i : 1 \leq i \leq j\}$.

Proof— ACL2 events

(auxiliary material)

```
(defun tuple-set-filter (S i)
  (cond ((endp S) NIL)
        ((and (consp (first S)) (<= (first (first S)) i))
         (cons (first S) (tuple-set-filter (rest S) i)))
        (T (tuple-set-filter (rest S) i))))

(defun tuple-set-projection (S)
  (cond ((endp S) NIL)
        ((consp (first S))
         (cons (rest (first S))
               (tuple-set-projection (rest S))))
        (T (tuple-set-projection (rest S)))))

(defun tuple-set->ordinal-partial-sum (k S i)
  (declare (xargs :measure
   (cons (1+ (nfix k))
         (nfix (- (tuple-set-max-first S) i)))))
  (cond ((or (not (natp k)) (not (natp i))) 0)
        ((zp k) 0)
        ((equal k 1)
         (tuple-set-min-first S))
        ((<= (tuple-set-max-first S) i)
         (o^ (omega)
             (o+ (tuple-set->ordinal-partial-sum
                  (1- k)
                  (tuple-set-projection S)
                  0)
                 1)))
        (T (o+
            (o^ (omega)
                (tuple-set->ordinal-partial-sum
                 (1- k)
                 (tuple-set-filter-projection S i)
                 0))
            (tuple-set->ordinal-partial-sum k S
                                             (1+ i))))))
```

Proof—ACL2 events

(auxiliary material)

```
(defthm map-lemma-3.7
  (implies (and (tuple-setp k A)
                (tuple-setp k B)
                (natp k)
                (< 1 k)
                (natp i)
                (natp j)
                (<= i j)
                (equal
                 (tuple-set->ordinal-partial-sum k A i)
                 (tuple-set->ordinal-partial-sum k B i)))
           (equal (equal (tuple-set->ordinal-partial-sum
                          (1- k)
                          (tuple-set-projection
                           (tuple-set-filter A j))
                          0)
                         (tuple-set->ordinal-partial-sum
                          (1- k)
                          (tuple-set-projection
                           (tuple-set-filter B j))
                          0))
                  T))

(defthm dixon-map-thm
  (implies (and (tuple-setp k S)
                (consp S)
                (natp k)
                (<= 1 k)
                (not (exists-partial-tuple-<=-set
                      k (rest S) (first S))))
           (o< (tuple-set->ordinal k S)
               (tuple-set->ordinal k (rest S)))))
```

Summary

- The need for a machine verified proof of Dickson's lemma lead to a new proof different from the classical ones.
- The effort motivated the development of an ordinal book for ACL2 which may well benefit other proof attempts as well.
- The proof enabled the verification of the termination of the Büchberger algorithm implemented in ACL2.
- Used 23 function definitions two of them ordinal related. Proved 80 theorems of which 26 was ordinal related (not counting the theorems imported from the ordinal book referenced).

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