Solving $\triangle = \square$

John R. Cowles
Department of Computer Science
University of Wyoming
Laramie, Wyoming
cowles@cs.uwyo.edu

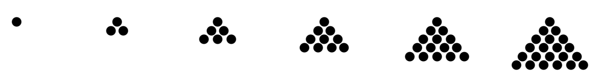
Ruben Gamboa
Department of Computer Science
University of Wyoming
Laramie, Wyoming
ruben@cs.uwyo.edu

ABSTRACT

For positive integer n, the **triangular number**, Δ_n, is defined by

$$\Delta_n = \sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n = \frac{n \cdot (n+1)}{2}.$$

Here are the first 6 triangular numbers:



$\Delta_1 = 1$ $\Delta_2 = 3$ $\Delta_3 = 6$ $\Delta_4 = 10$ $\Delta_5 = 15$ $\Delta_6 = 21$

Problem. Find triangular numbers that are also squares.
That is, find positive integers, n and k, such that

$$\frac{n \cdot (n+1)}{2} = k^2$$

or

$$n \cdot (n+1) = 2 \cdot k^2.$$

Clearly $\Delta_1 = 1$ is a square. Are there other square triangular numbers? Are there infinitely many square triangular numbers? These questions are answered below and the answers have been formally verfied using ACL2.

Categories and Subject Descriptors

F.4.1 [**Mathematical Logic and Formal Languages**]: Mathematical Logic—*Mechanical theorem proving*

General Terms

Verification

Keywords

Boyer-Moore logic, ACL2, Pell's equation, triangular numbers

.

1. REFORMULATE THE PROBLEM

These first three lemmas show how to transform the original problem into one that has been studied for a long time and has a well understood solution [2, pages 88–100]. The proofs use elementary algebra.

Lemma 1. If n and k are positive integers such that

$$n \cdot (n+1) = 2 \cdot k^2,$$

then $x = 2 \cdot n + 1$ and $y = 2 \cdot k$ are positive integers such that

$$x^2 - 2 \cdot y^2 = 1.$$

Moreover, $x \geq 3$ is odd and $y \geq 2$ is even.

Lemma 2. If x and y are positive integers such that

$$x^2 - 2 \cdot y^2 = 1,$$

then x is odd and y is even. Moreover, $x \geq 3$ and $y \geq 2$.

Lemma 3. If x and y are positive integers such that

$$x^2 - 2 \cdot y^2 = 1,$$

then $n = \frac{x-1}{2}$ and $k = \frac{y}{2}$ are positive integers such that

$$n \cdot (n+1) = 2 \cdot k^2.$$

For squarefree positive integers D, the equation

$$x^2 - D \cdot y^2 = 1$$

is usually called Pell's equation. However, John Pell (1610–1685), an English Mathematician, made no contribution to the study of this equation [1, page 246].

The obvious solution, $n = 1, k = 1$, to our original problem corresponds to the solution, $x = 3, y = 2$, of the Pell equation

$$x^2 - 2 \cdot y^2 = 1.$$

2. GENERATE MANY SOLUTIONS

This lemma shows how to construct new solutions to our particular Pell equation from known solutions. Once more the proof depends only on elementary algebra.

Lemma 4. If a, b, c, and d are positive integers such that

$$a^2 - 2 \cdot b^2 = 1,$$

and

$$c^2 - 2 \cdot d^2 = 1,$$

then $x = a \cdot c + 2 \cdot b \cdot d$ and $y = a \cdot d + b \cdot c$ are positive integers such that

$$x^2 - 2 \cdot y^2 = 1.$$

In the above lemma, the two given solutions, a, b and c, d, need not be distinct. That is, using $a = c$ and $b = d$ in the construction of x, y yields a new solution. Thus, starting with one known solution, $a = 3, b = 2$, many other solutions can be generated:

Definition 1. For each positive integer, j, recursively define (x_j, y_j) by

$$\begin{aligned}(x_1, y_1) &= (3, 2)\\ (x_{j+1}, y_{j+1}) &= (x_1 \cdot x_j + 2 \cdot y_1 \cdot y_j, x_1 \cdot y_j + x_j \cdot y_1)\end{aligned}$$

Examples.

j	1	2	3	4	5	6
x_j	3	17	99	577	3363	19601
y_j	2	12	70	408	2378	13860

j	7	8	9
x_j	114243	665857	3880899
y_j	80782	470832	2744210

The next theorem is proved by mathematical induction on j, using Lemma 4.

Theorem 1. For each positive integer j,

$$x_j^2 - 2 \cdot y_j^2 = 1.$$

3. NO OTHER SOLUTIONS

In fact, the solutions given by Definition 1 are all the positive integer solutions:

This lemma constructs another new solution, (a, b), to our Pell equation from a known solution, (x, y). This time the new solution is "smaller" than the old solution in the sense that $b < y$. The proof only requires elementary algebra.

Lemma 5. If x and y are positive integers such that $y > 2$ and

$$x^2 - 2 \cdot y^2 = 1,$$

then $a = 3 \cdot x - 4 \cdot y$ and $b = -2 \cdot x + 3 \cdot y$ are positive integers such that

$$a^2 - 2 \cdot b^2 = 1.$$

Moreover, $b < y$.

The proof of this next theorem is most interesting, both mathematically and as a formalization challenge in ACL2.

Theorem 2. If x and y are positive integers such that

$$x^2 - 2 \cdot y^2 = 1,$$

then for some positive integer j, $(x, y) = (x_j, y_j)$.

Proof. By contradiction. Suppose there are positive integers x and y such that for all positive integers j, $(x, y) \neq (x_j, y_j)$ and

$$x^2 - 2 \cdot y^2 = 1.$$

Since the positive integers are well-ordered, pick such an x and y with y as small as possible. Since $y \neq y_j$ for any j, then $y > 2$. By Lemma 5, $a = 3 \cdot x - 4 \cdot y$ and $b = -2 \cdot x + 3 \cdot y$ are positive integers that satisfy $b < y$ and

$$a^2 - 2 \cdot b^2 = 1.$$

Since y was chosen as small as possible and $b < y$, there must be a j such that $(a, b) = (x_j, y_j)$. By Definition 1,

$$\begin{aligned}(x_{j+1}, y_{j+1}) &= (x_1 \cdot x_j + 2 \cdot y_1 \cdot y_j,\\ &\quad\; x_1 \cdot y_j + x_j \cdot y_1)\\ &= (3 \cdot a + 2 \cdot 2 \cdot b, 3 \cdot b + a \cdot 2)\\ &= (x, y).\end{aligned}$$

This contradicts the above supposition.

Now the solutions of the Pell equation may be translated into solutions of the original problem.

Definition 2. For each positive integer, j, define (n_j, k_j) by

$$(n_j, k_j) = \left(\frac{x_j - 1}{2}, \frac{y_j}{2}\right).$$

Examples.

j	1	2	3	4	5	6
n_j	1	8	49	288	1681	9800
k_j	1	6	35	204	1189	6930

j	7	8	9
n_j	57121	332928	1940449
k_j	40391	235416	1372105

Combining the results of Lemma 1, Lemma 3, Theorem 1, and Theorem 2 yields all the positive integer solutions of the original problem.

Theorem 3. For each positive integer j,

$$\frac{n_j \cdot (n_j + 1)}{2} = k_j^2.$$

If n and k are positive integers such that

$$\frac{n \cdot (n + 1)}{2} = k^2,$$

then for some positive integer j, $(n, k) = (n_j, k_j)$.

4. THE PROOFS IN ACL2

Proofs, for Lemma 1, Lemma 3, Lemma 4, Theorem 1, Theorem 3, require only the use of a good arithmetic book available with the ACL2 distribution. Proofs, for Lemma 2, Lemma 5, requiring only elementary algebra, proceed with many tiny steps explicitely spelled out in their own `defthm`'s.

From the point of view of the ACL2 translation, the most interesting step in the proof of Theorem 2 is this: Given at least one pair, (x, y), of positive integers such that $x^2 - 2 \cdot y^2 = 1$ and for all positive integers j, $(x, y) \neq (x_j, y_j)$; we wish to locate the pair, (q, r), of positive integers, with r as small as possible, such that $q^2 - 2 \cdot r^2 = 1$ and for all positive integers j, $(q, r) \neq (x_j, y_j)$. The desired (q, r) is found by an ACL2 function that returns the first pair in this list, of y pairs,

$$(1,1),\ldots,\left(q=\left\lfloor\sqrt{2r^2+1}\right\rfloor,r\right),\ldots,\left(x=\left\lfloor\sqrt{2y^2+1}\right\rfloor,y\right),$$

that satisfies both $q^2-2\cdot r^2=1$ and for all positive integers j, $(q,r)\neq(x_j,y_j)$.

5. CONCLUSION

An algorithm, described in Definition 1 and Definition 2, is presented that enumerates all pairs, (n,k), of positive integers such that $\Delta_n=k^2$. ACL2 is used to verify that the algorithm is correct. The ACL2 proofs and definitions are included in the supporting materials for this Workshop and future distributions of ACL2.

6. REFERENCES

[1] C. V. Eynden. *Elementary Number Theory*. Random House, 1987.

[2] W. Sierpiński. *Elementary Theory of Numbers*. PWN–Polish Scientific Publishers, 1987. Second English edition revised and enlarged by A. Schinzel.

APPENDIX

A. MORE EFFICIENT GENERATION OF SOLUTIONS

Recall that the pairs (x_j,y_j), defined in Definition 1, are all the positive integer solutions to $x^2-2\cdot y^2=1$.

Definition 1. For each positive integer, j, recursively define (x_j,y_j) by

$$\begin{aligned}(x_1,y_1)&=(3,2)\\(x_{j+1},y_{j+1})&=(x_1\cdot x_j+2\cdot y_1\cdot y_j,x_1\cdot y_j+x_j\cdot y_1)\end{aligned}$$

A straightforward implementation of this definition uses a linear number, in j, of arithmetic operations to compute each (x_j,y_j).

Clever use of Lemma 4, produces an alternative sequence of pairs $(\bar{x}_j,\bar{y}_j)$ that uses a logarithmic number, in j, of arithmetic operations to compute each pair:

Definition 1A. For each positive integer, j, recursively define $(\bar{x}_j,\bar{y}_j)$ by

$$\begin{aligned}(\bar{x}_1,\bar{y}_1)&=(3,2)\\(\bar{x}_{2j},\bar{y}_{2j})&=(\bar{x}_j^2+2\cdot\bar{y}_j^2,2\cdot\bar{x}_j\cdot\bar{y}_j)\\(\bar{x}_{2j+1},\bar{y}_{2j+1})&=(\bar{x}_1\cdot(\bar{x}_j^2+2\cdot\bar{y}_j^2)+4\cdot\bar{y}_1\cdot\bar{x}_j\cdot\bar{y}_j,\\&\qquad 2\cdot\bar{x}_1\cdot\bar{x}_j\cdot\bar{y}_j+\bar{y}_1\cdot(\bar{x}_j^2+2\cdot\bar{y}_j^2))\end{aligned}$$

The two sequences coincide:

Theorem 4. For each positive integer j,

$$(x_j,y_j)=(\bar{x}_j,\bar{y}_j).$$