A Mechanized Proof of the Curve Length of a Rectifiable Curve

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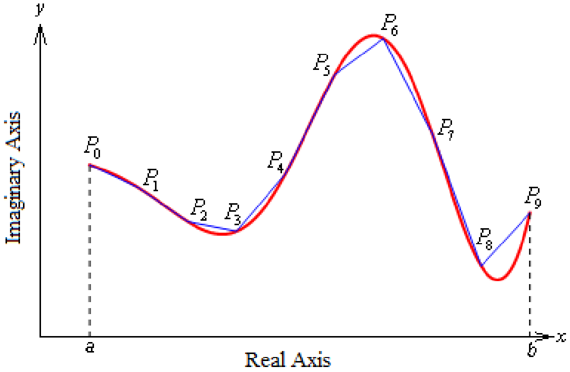
ACL2 Workshop 2017

May 23, 2017

Theory



$$L \approx \sum_{i=1}^{n} | P_i - P_{i-1} |$$

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$$= \int_{t_0}^{t_n} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Continuously Differentiable curve

```
(encapsulate
  ((c (x) t)
   (c-derivative (x) t))
  ;; Our witness continuous function is the identit
  (local (defun c(x) x))
  (local
   (defun c-derivative (x) (declare (ignore x)) 1
     ))

   ; (i-close (/ (- (c x) (c y)) (- x y))
                         ;(c-derivative x))

   ; (i-close (c-derivative x) (c-derivative y))
  )
```

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- Sum of 2 continuous functions is continuous. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ is continuous

Square root of a continuous function is Continuous

```
(implies (and (realp y1)
              (realp y2)
              (i-limited y1)
              (i-limited y2)
              (>= y1 0)
              (>= y2 0)
              (not (i-close y1 y2)))
         (not (= (standard-part (square y1))
                 (standard-part (square y2)))))

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```

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```
(defthmd root-close-f
  (implies
   (and (standardp x1)
        (realp x1)
        (realp x2)
        (>= x1 0)
        (>= x2 0)
        (i-close x1 x2))
   (i-close (acl2-sqrt x1) (acl2-sqrt x2)))
   ;hints omitted
   )
```

$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is continuous

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- Thus as $n \to \infty, \Delta t$ is infinitely small and riemann sum is equal to $\int_{t_0}^{t_n} h(t)dt$

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Thus by using above proof length of $f(t)$ is equal to

$$\int_0^{2\pi} |g(t)| \, dt$$

Applying second Fundamental Theorem of Calculus

$g(t) = r * (-\sin t + i * \cos t)$

$|g(t)| = r; \qquad \therefore \int_0^{2\pi} |g(t)|\, dt = \int_0^{2\pi} r \;\, dt$

Let, $h(t) = r * t, \qquad h'(t) = \mid g(t) \mid$

$\therefore$ Using second fundamental theorem of calculus

$$\int_0^{2\pi} \mid g(t) \mid dt = h(2\pi) - h(0) = r * 2 * \pi$$