Cellular Automata Surviving k Steps

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May 2025

Surviving k Steps

1. Problem posed by Wolfram in 2024.

[1] writings.stephenwolfram.com posts dated May 3, 2024; August 22, 2024; December 5, 2024; February 3, 2025.

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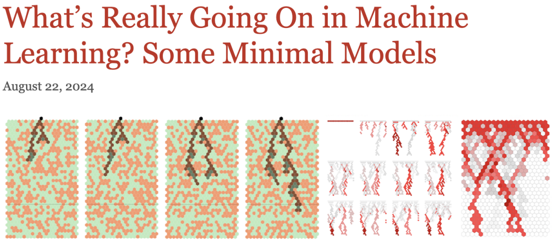

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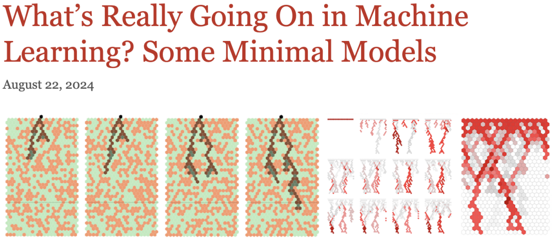

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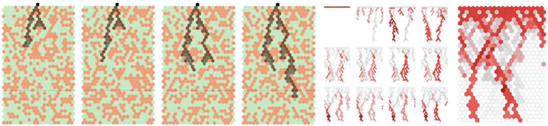
August 22, 2024



3. Wolfram takes an empirical view, we take a mathematical one.
4. Our work: $> 99\%$ solved.

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Elementary Cellular Automata

1. Functions from $\{\blacksquare, \square\}^3 \to \{\blacksquare, \square\}$.

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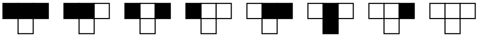
2. Let λ_i be the ith one ("Wolfram Code" enumeration).
3. For example, λ_4's truth table

λ_4

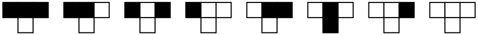


$$\lambda_4(\blacksquare, \blacksquare, \blacksquare) = \square, \quad \lambda_4(\square, \blacksquare, \square) = \blacksquare, \quad \ldots$$

Growing λ_4 (visually)

$x^{(j)} =$ 

λ_4



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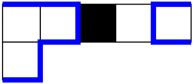

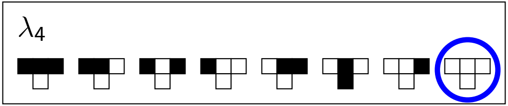

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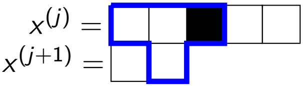

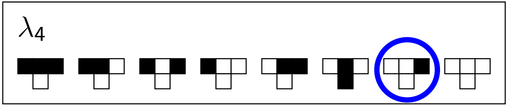

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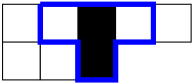

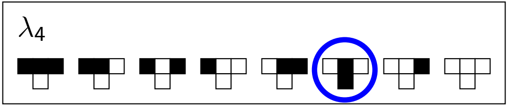

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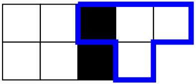

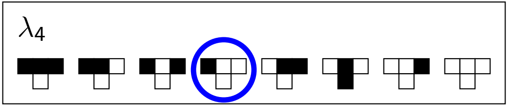

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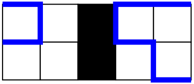

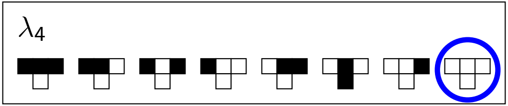

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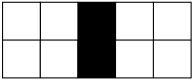

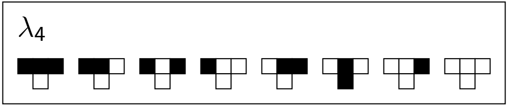

Growing λ_4

jth state $x^{(j)} \in \{\blacksquare, \square\}^w$, next state $x^{(j+1)}$,

$$x_i^{(j+1)} = \lambda_4(x_{i-1 \bmod w}^{(j)}, x_i^{(j)}, x_{i+1 \bmod w}^{(j)})$$

Inhomogeneous Cellular Automata

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2. Formally, use a *rule array* $a : \mathbb{N}^2 \to \{\lambda_{i_1}, \ldots, \lambda_{i_t}\}$,

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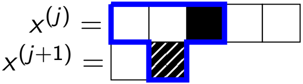
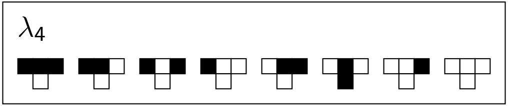
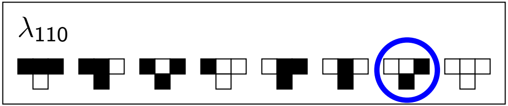
Growing Inhomogeneous Cellular Automata (visually)



$$a(1, j) = \lambda_{110}$$





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3. **What can we synthesize with appropriate choice of a?**

Figure: An inhomogeneous cellular automata using λ_{126} and λ_{225}.

Inhomogeneous Cellular Automata

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[2] *A survey of cellular automata: types, dynamics, non-uniformity and applications*, Kamalika Bhattacharjee *et. al.*

Inhomogeneous Cellular Automata

1. Inhomogeneous Cellular Automata are very expressive[2].
2. Our work looks at the Survive-k-Steps problem posed by Wolfram in 2024.

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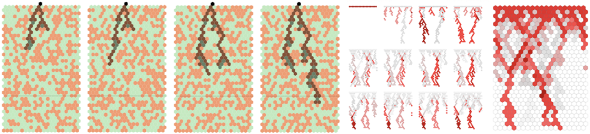

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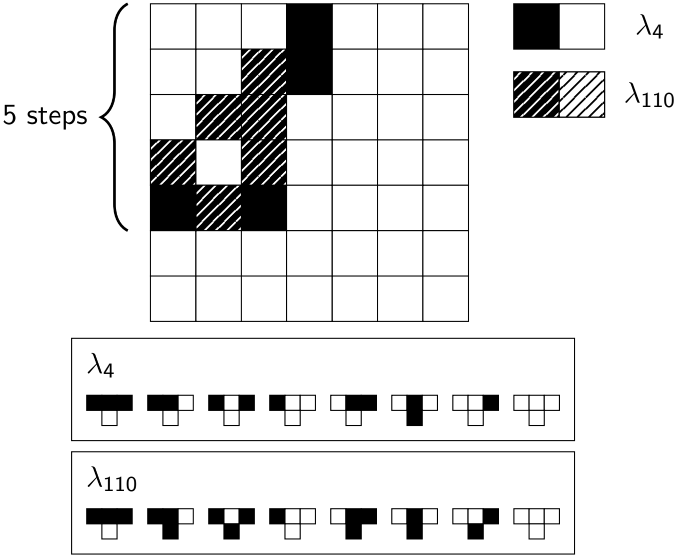
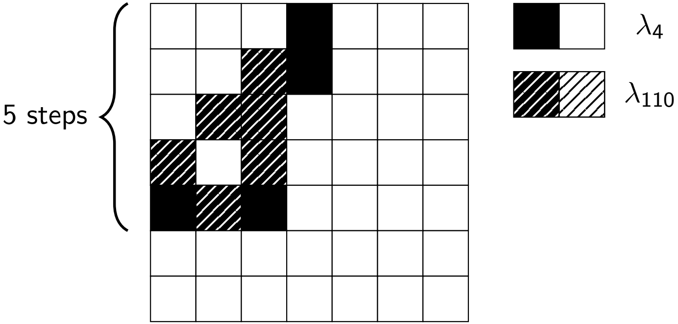
Survive-5-Steps (visually)



Survive-5-Steps (visually)



This proves λ_4, λ_{110} can Survive-5-Steps, but which k are solvable in general?

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then, λ_i and λ_j can solve Survive-k-Steps .

Research Question

For any i, j, is the set

$$\{k \mid \lambda_i, \lambda_j \text{ solve Survive-}k\text{-Steps}\}$$

finite?

Evidence of Structure

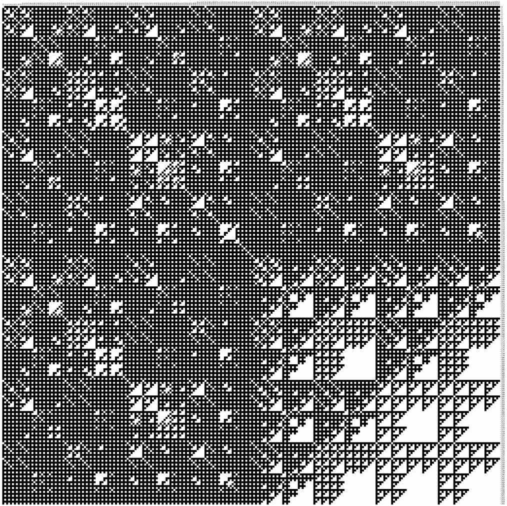
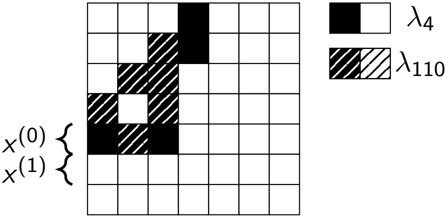


Figure: ■ at position (i, j) means Survive-50-Steps on a width 31 board is possible with λ_i and λ_j. Top-left is $(0, 0)$, bottom right is $(255, 255)$.

The ■-to-□ Transition

To Survive-k-Steps λ_i,λ_j need to “die”.



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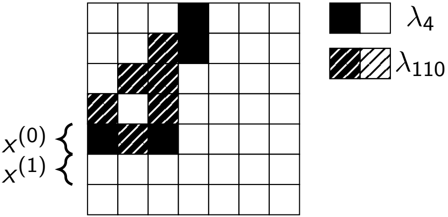

"die", i.e. $\exists x^{(0)}$, $\blacksquare \in x \wedge \blacksquare \notin x^{(1)}$,i.e. the ■-to-□ Transition.

When is ■-to-□ impossible?

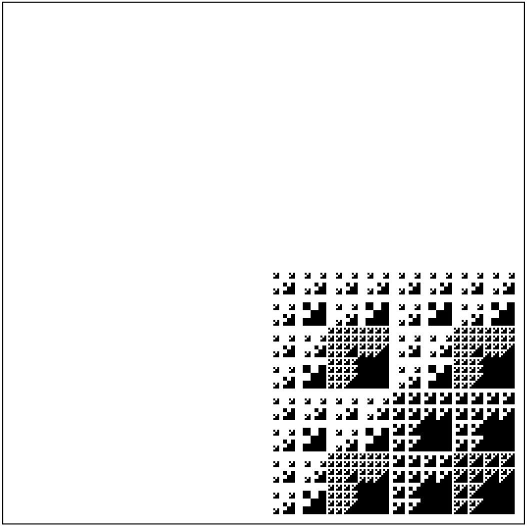

Figure: ■ at position (i, j) means λ_i, λ_j cannot do the ■-to-□ transition on a width 31 board. Top-left is $(0, 0)$, bottom right is $(255, 255)$.

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Let's prove it.

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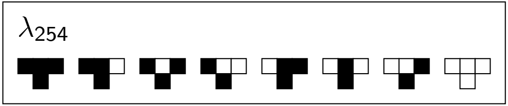

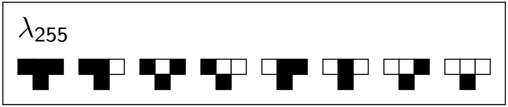

   cannot transition from $\blacksquare$-to-$\square$.

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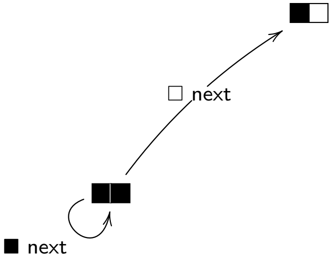

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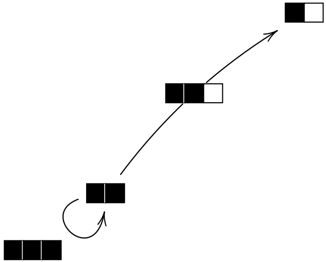

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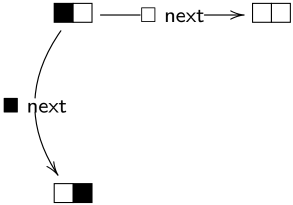

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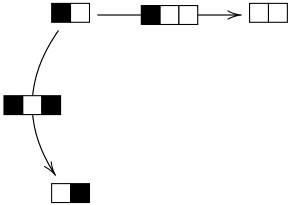

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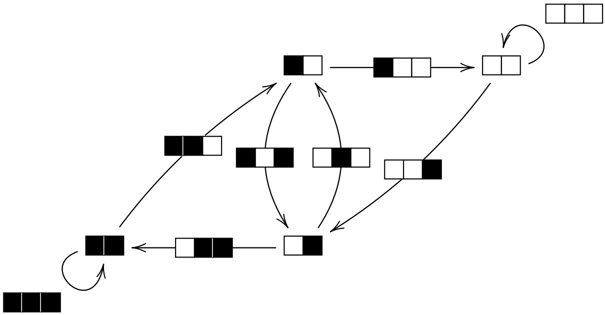

This is a De Bruijn graph.

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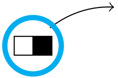

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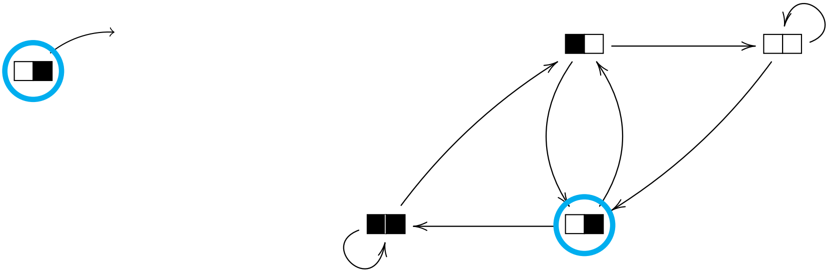

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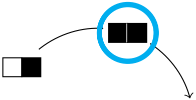

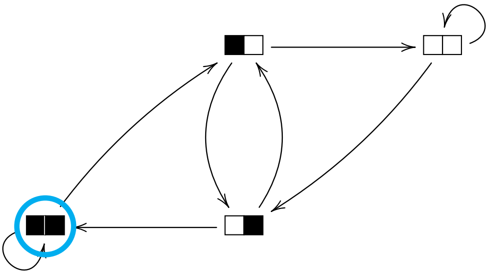

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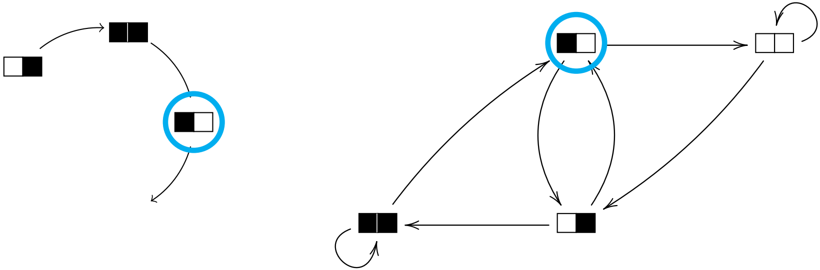

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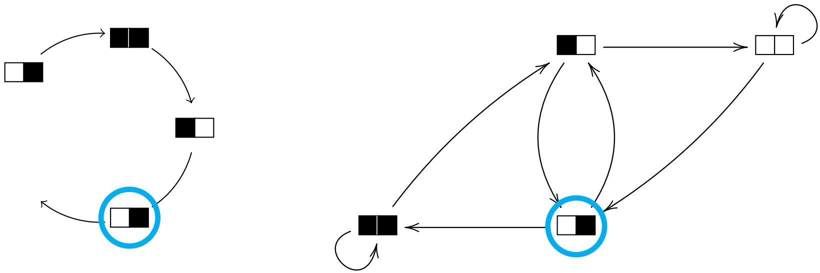

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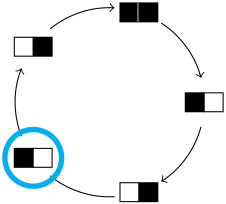

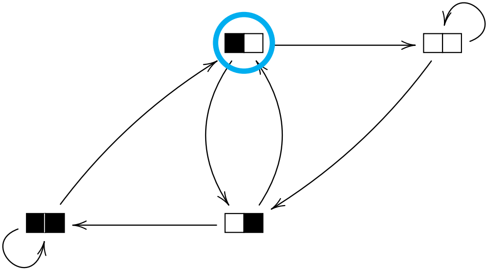

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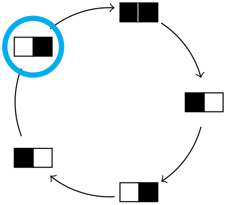

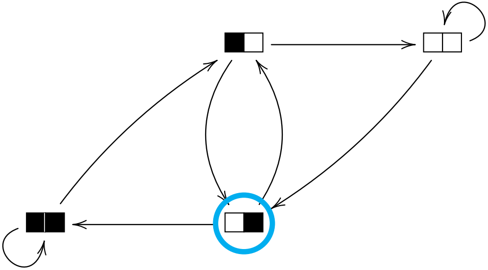

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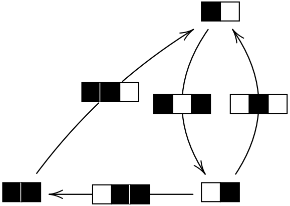

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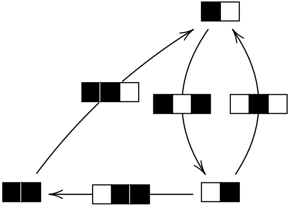
$$x = □■■□■.$$

Corresponds to



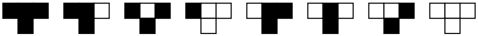
Corresponds to

■□■, □■■, ■■□, and □■□.

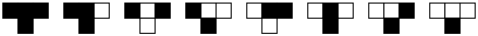
■-to-□ with λ_{238}, λ_{215}

Let's connect our graph with ability to transition.

λ_{238}



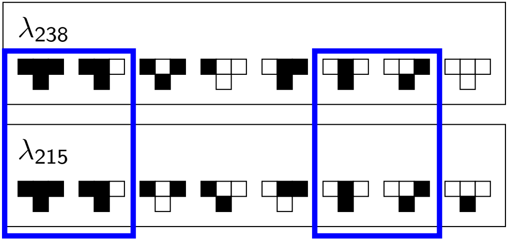
λ_{215}



Can these automata do the ■-to-□ transition?

■-to-□ with λ_{238}, λ_{215}

Both output ■ on ■■■, ■■□, □■□, □□■.



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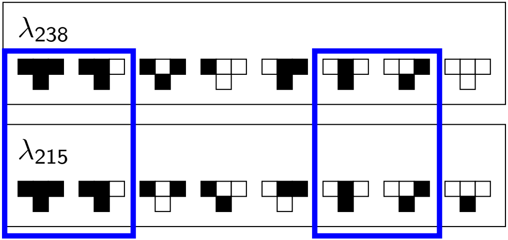
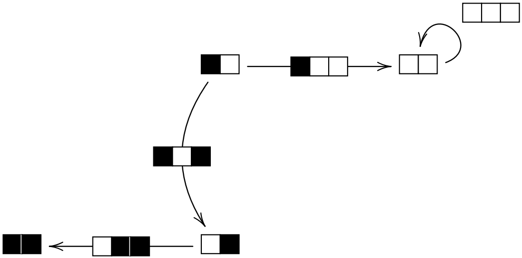


Let's remove those edges and see if we can still make an $x^{(0)}$.

■-to-□ with λ_{238}, λ_{215}

λ_{238}, λ_{215}'s graph:



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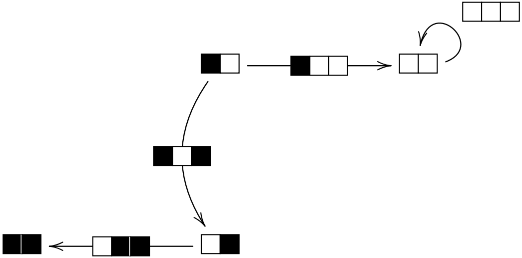
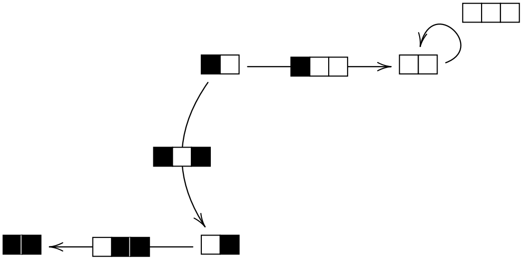


There are no cycles with a ■.

■-to-□ with $\lambda_{238}, \lambda_{215}$

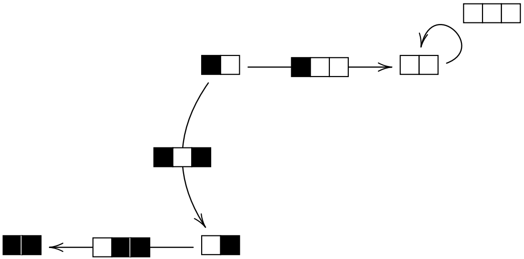
$\lambda_{238}, \lambda_{215}$'s graph:



So there are no states containing a ■ that $\lambda_{238}, \lambda_{215}$ aren't forced to output a ■ on.

■-to-□ with λ_{238}, λ_{215}

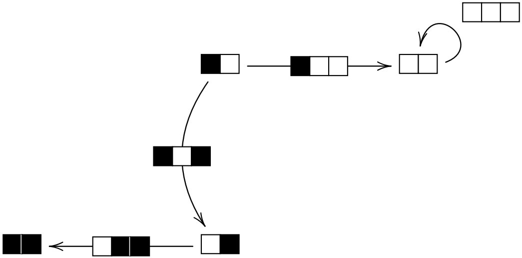
λ_{238}, λ_{215}'s graph:



So using λ_{238}, λ_{215} the ■-to-□ transition is impossible.

■-to-□ with λ_{238}, λ_{215}

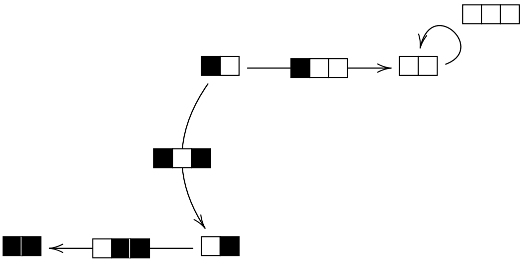
λ_{238}, λ_{215}'s graph:



So λ_{238}, λ_{215} cannot "die".

■-to-□ with λ_{238}, λ_{215}

λ_{238}, λ_{215}'s graph:



So λ_{238}, λ_{215} cannot Survive k Steps for any k.

When is ■-to-□ impossible?

Putting this together.

If we prune all edges from our graph labeled with a pattern both λ_i, λ_j output ■ on

When is ■-to-□ impossible?

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If we prune all edges from our graph labeled with a pattern both λ_i, λ_j output ■ on and there is cycle passing through a node with a ■,

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Putting this together.
If we prune all edges from our graph labeled with a pattern both λ_i, λ_j output ■ on and there is cycle passing through a node with a ■, that cycle corresponds to $x^{(0)}$ s.t. $\blacksquare \in x^{(0)}$ and λ_i,λ_j can be composed to make $x^{(1)}$ all-□.

When is $\blacksquare$-to-$\square$ impossible?

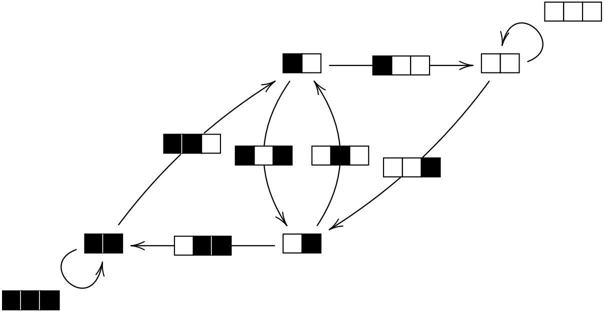
Putting this together.
If we prune all edges from our graph labeled with a pattern both λ_i, λ_j output $\blacksquare$ on and there is cycle passing through a node with a $\blacksquare$, that cycle corresponds to $x^{(0)}$ s.t. $\blacksquare \in x^{(0)}$ and λ_i,λ_j can be composed to make $x^{(1)}$ all-$\square$. I.e the. $\blacksquare$-to-$\square$ transition is possible.

When is ■-to-□ impossible?

Putting this together.
If we prune all edges from our graph labeled with a pattern both λ_i, λ_j output ■ on and there is cycle passing through a node with a ■, that cycle corresponds to $x^{(0)}$ s.t. $\blacksquare \in x^{(0)}$ and λ_i,λ_j can be composed to make $x^{(1)}$ all-□. I.e the. ■-to-□ transition is possible. **Q.E.D.**

Automating This

In the general case, checking for length-w cycles can be done efficiently.



Hint: Every cycle is a linear combination of simple cycles.

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